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Ye. S. VENTTSEL

ELEMENTS OF GAME THEORY

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ПОПУЛЯРНЫЕ ЛЕКЦИИ ПО МАТЕМАТИКЕ

Е. С. Вентцель

ЭЛЕМЕНТЫ ТЕОРИИ ИГР

ИЗДАТЕЛЬСТВО «НАУКА» МОСКВА

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ELEMENTS OF GAME THEORY

Translated from the Russian
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MIR PUBLISHERS
MOSCOW

First published 1980

На английском языке

Contents

1. The Subject-matter of Game Theory. Basic Concepts

When solving practical (economic, military or other) problems one often has to analyse situations in which there are two, or more quarrelling parties pursuing conflicting objectives, and where the outcome of each action of one party depends on the opponent's choice of a course of action. Such situations will be called "conflict situations".

One can cite numerous examples of conflict situations from various practical situations. All situations arising in the course of military action are conflict situations: each of the contending parties takes every available measure to prevent the opponent from succeeding. Conflict situations also arise in choosing a weapon or a mode of its combat use, and in general, in planning military operations; every decision in this area must be made assuming that the opponent's action will be the least favourable one. A number of economic situations (especially those where there is free competition) are conflict situations. The contending parties here are firms, industrial enterprises, etc.

The need to analyse situations of this kind has brought about the development of special mathematical methods. The theory of games is in fact a mathematical theory of conflict situations. The aim of the theory is to elaborate recommendations for each of the opponents to act rationally in the course of a conflict situation.

All conflict situations that occur in practice are very complicated and their analysis is hampered by many attendant factors. For a mathematical analysis of a situation to be possible, it is necessary to disengage oneself from these secondary factors and construct a simplified, formalized model of the situation. Such a model will be called a "game".

A game differs from a real conflict situation in that it is played according to definite *rules*. Man has long used such formalized models of conflict situations – *games* in the literal sense of the word, for example chess, checkers, card games and so on. Each of these games takes the form of a contest proceeding according to certain rules and ending in one or another player's "victory" (or gain).

Such formally regulated, artificially arranged games provide the most suitable material for illustrating and learning the basic concepts of game theory. The terminology borrowed from such games is used in the analysis of other conflict situations as well;

the convention has been adopted of referring to the parties taking part in them as "players" and to the outcome of an encounter as a party's "gain" or "payoff".

A game may be clash of interests of two or more opponents; in the first case the game is called a two-person game and in the second it is called a multiperson game. The participants of a multiperson game may, in the course of the game, form coalitions, constant or temporary. If there are two constant coalitions in a multiperson game, it becomes a two-person game. In practice the most important games are two-person games. We shall confine ourselves to these games only.

We shall begin our presentation of elementary game theory by formulating some basic concepts. We shall consider the two-person game in which two players, A and B, with opposing interests take part. By a "game" we shall mean an arrangement consisting of a number of actions taken by parties A and B. For a game to be treated mathematically it is necessary to formulate the *rules of the game* exactly. The "rules of a game" are understood to be a set of conditions which regulate the conceivable alternatives for each party's course of action, the amount of information each party has about the other party's behaviour, the sequence of alternating the "moves" (individual decisions made in the course of the game), and the *result* or *outcome* to which the given totality of moves leads. This result (gain or loss) does not always have a quantitative expression, but it is usually possible to express the result by a certain number by establishing some measuring scale. For example, it might be agreed in chess to assign the value $+1$ to a victory, the value -1 to a defeat, and the value 0 to a draw.

A game is called a *zero-sum* game if one player gains what the other loses, i. e. if the sum of both parties' gains is equal to zero. In a zero-sum game the players' interests are completely opposed. We shall consider only zero-sum games in the following.

Since in a zero-sum game one player's gain is the other player's gain taken with the opposite sign, it is obvious that in analysing such a game it is possible to consider just one player's gain. Let us take player A, for example. For the sake of convenience, in what follows we shall arbitrarily refer to party A as "we" and to party B as "the opponent".

Party A ("we") will always be regarded as the winner and party B ("the opponent") as the loser. Evidently this formal condition does not imply any real advantage to the first player. It can easily be seen that it can be replaced by the opposite condition if the sign of the gain is reversed.

We shall consider the development of a game in time as a series of successive steps or "moves". In game theory a *move* is a choice of an alternative from the alternatives that are allowed by the rules of the game. Moves can be classified as *personal* or *chance* moves.

A *personal move* is a player's deliberate choice of one of the moves possible in the given situation, and its realization.

An example of a personal move is a move in a game of chess. In making his move, a player makes a deliberate choice among the alternatives possible for a given disposition of pieces on the chessboard.

The set of possible alternatives is stipulated for each personal move by the rules of the game and depends on the totality of both parties' previous moves.

A *chance move* is a choice among a number of possibilities which is realized not by the player's decision but by some random device (the tossing of a coin or a dice, the shuffling and dealing of cards, etc.). For example, dealing the first card to a bridge player is a chance move with 52 equally possible alternatives.

For a game to be mathematically definite, the rules must indicate for each chance move the *probability distribution* of the possible outcomes.

Some games may contain only chance moves (the so-called games of pure chance) or only personal moves (chess, checkers). Most card games are of mixed type, i. e. they consist of both chance and personal moves.

Games are classified not only according to the nature of the moves (into personal and chance moves), but also according to the nature and amount of information available to either player concerning the other's actions. A special class of games is formed by "games with perfect information". A *game with perfect information* is a game in which either player knows at each move the results of all the previous moves, both personal and chance. Examples of games with perfect information are chess, checkers, and the well-known game of "noughts-and-crosses".

Most of the games of practical importance are not games with perfect information since the lack of information about the opponent's actions is usually an essential element of conflict situations.

One of the basic concepts of game theory is the concept of a "strategy".

A *strategy* for player is a set of rules unambiguously determining the choice of every personal move of the player, depending on the situation that has arisen in the course of the game.

The concept of strategy should be explained in more detail.

A decision (choice) for each personal move is usually made by the player in the course of the game itself depending on the particular situation that has arisen. Theoretically, the situation will not be altered, however, if we imagine that all the decisions are made by the player *in advance*. To that end the player would have to make a list of all situations that might occur in the course of the game beforehand and foresee his decision for each of them. This is possible in principle (if not in practice) for any game. If such a system of decisions is adopted, it means that the player has chosen a definite *strategy*.

A player who has chosen a strategy may now abstain from taking part in the game personally and substitute for his participation a list of rules to be applied for him by some disinterested person (a referee). The strategy may also be given to an automatic machine in the form of a certain programme. It is in this way that modern electronic computers play chess.

For the concept of "strategy" to have sense, a game must have personal moves; there are no strategies in games comprising only chance moves.

Games are classified into "finite" and "infinite" ones depending on the number of possible strategies.

A game is said to be *finite* if either player has only a finite number of strategies.

A finite game in which player A has m strategies and player B n strategies is called an $m \times n$ game.

Consider an $m \times n$ game between two players, A and B ("we" and "our opponent").

We denote our strategies by $A_1, A_2, \ldots, A_m$, and our opponent's strategies by $B_1, B_2, \ldots, B_n$.

Suppose that either party has chosen a definite strategy; let it be A_i in our case, and B_j in our opponent's.

If the game contains only personal moves, then the choice of strategies A_i, B_j unambiguously determines the outcome of the game – our gain (payoff). We denote it by a_{ij}.

If the game consists of chance moves as well as personal ones, then the gain for a pair of strategies A_i and B_j is a random quantity depending on the outcomes of all chance moves. In this case the natural estimate of the gain expected is its *mean value* (mathematical expectation). We shall denote by the same symbol a_{ij} both the gain (payoff) itself (in games without chance moves) and its mean value (in games with chance moves).

Suppose we know the value a_{ij} of the gain or payoff (or the average values) for each pair of strategies. The values a_{ij} can be

written in the form of a rectangular array (or matrix) the rows of which correspond to our strategies (A_i) and the columns to our opponent's strategies (B_j). This array is called a *gain* or *payoff* matrix, or simply the *matrix of the game* or *game matrix.*

The matrix of an $m \times n$ game is of the form:

A \ B	B_1	B_2		B_n
A_1	a_{11}	a_{12}		a_{1n}
A_2	a_{21}	a_{22}		a_{2n}
. .	. .	. .	. .	. .
A_m	a_{m1}	a_{m2}		a_{mn}

We shall briefly denote a game matrix by $\|a_{ij}\|$.

Consider a few elementary examples of games.

EXAMPLE 1. Two players, A and B, without looking at each other, each places a coin face up on a table showing either heads or tails as they like. If both choose the same side (either heads or tails), then player A takes both coins; otherwise they are taken by player B. Analyse the game and construct its matrix.

SOLUTION. The game contains only two moves, our move and our opponent's move; both are personal. It is not a game with perfect information since at the moment a move is made the player who makes it does not know what the other player will do.

Since either player has only one personal move, a player's strategy is a choice for this single personal move.

There are two strategies for us: choosing heads, A_1, and choosing tails, A_2; there are the same two strategies for our opponent: heads, B_1, and tails, B_2. So this is a 2×2 game. Let a gain of the coin count $+1$. The game matrix is given below.

This game, simple as it is, may help us to understand some essential ideas of game theory.

First assume that the game is played only once. Then it is evidently useless to speak of any "strategies" for the players

B / A	B_1 (heads)	B_2 (tails)
A_1 (heads)	1	-1
A_2 (tails)	-1	1

being more clever than the others. Either player may equally reasonably make either decision. When the game is repeated, however, the situation changes.

Indeed, assume that we (player A) have chosen some strategy (say A_1) and are keeping to it. Then from our initial moves our opponent will guess what our strategy is and respond to it in the manner least advantageous to us, i. e. he will choose tails. It is clearly not advantageous for us to always play only one of our strategies; in order not to lose we must sometimes choose heads and sometimes tails. However, if we alternate heads and tails in any definite order (for example, one after the other), our opponent may also guess this and counter our strategy in the worst manner for us. Evidently a reliable method which would guarantee that our opponent is not aware of our strategy is to arrange our choice of each move in such a way that we do not know it in advance ourselves (this could be ensured by tossing a coin, for instance). Thus, through an intuitive argument we have approached one of the essential concepts of game theory, that of a "mixed strategy", i. e. a strategy in which "pure" strategies – A_1 and A_2 in this case – are alternated randomly with certain frequencies. In this example, it is known in advance from symmetry considerations that strategies A_1 and A_2 should be alternated with the same frequency; in more complicated games, the decision may be far from being trivial.

EXAMPLE 2. Players A and B each write down simultaneously and independently of each other, one of the three numbers: 1, 2, or 3.

If the sum of the numbers they have written down is even, then B pays A that sum in dollars; if the sum is odd, then, on the contrary, A pays that sum to B. Analyse the game and construct its matrix.

SOLUTION. The game consists of two moves, both of which are personal. We (A) have three strategies: writing down 1, A_1; writing down 2, A_2; and writing down 3, A_3. Our opponent (B) has the same three strategies. This is a 3×3 game with the matrix given below.

B / A	B_1	B_2	B_3
A_1	2	-3	4
A_2	-3	4	-5
A_3	4	-5	6

Evidently our opponent can, as in the previous case, respond to any strategy chosen in the way which is worst for us. Indeed, if we choose strategy A_1, for instance, our opponent will always counter it with strategy B_2; strategy A_2 will always be countered with strategy B_3 and strategy A_3 with strategy B_2; thus any choice of a definite strategy will inevitably lead us to a loss.* The solution of this game (i. e. the set of the most advantageous strategies for both players) is given in Section 5.

EXAMPLE 3. We have three kinds of weapon at our disposal, A_1, A_2, and A_3; the enemy has three kinds of aircraft, B_1, B_2, and B_3. Our goal is to hit an aircraft, while the enemy's goal is to keep it unhit. When armament A_1 is used, aircraft B_1, B_2, and B_3 are hit with probabilities 0.9, 0.4, and 0.2, respectively; when armament A_2 is used, they are hit with probabilities 0.3, 0.6, and 0.8; and when armament A_3 is used, with probabilities 0.5, 0.7 and 0.2. Formulate the game in terms of game theory.

SOLUTION. The situation can be regarded as a 3×3 game with two personal moves and one chance move. Our personal move is to choose a kind of a weapon; the enemy's personal move is to choose an aircraft to take part in the combat. The chance move is to select the weapon; this move may or may not end with hitting the aircraft. Our gain is unity if the aircraft is hit, otherwise it is zero. Our strategies are the three alternative weapons; the enemy's strategies are the three alternative aircraft.

The mean value of the gain in each of the specified pairs of strategies is just the probability of hitting a given aircraft with a given weapon. The game matrix is given below.

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* One should not forget, however, that our opponent's position is as bad as ours.

B \ A	B_1	B_2	B_3
A_1	0.9	0.4	0.2
A_2	0.3	0.6	0.8
A_3	0.5	0.7	0.2

The aim of game theory is to work out recommendations for the player' rational behaviour in conflict situations, i. e. to determine an "optimal strategy" for each player.

In game theory, an *optimal strategy* for a player is a strategy which, when repeated many times, assures him the maximum possible average gain (or, which amounts to the same thing, the minimum possible average loss). The argument for the choice of this strategy is based on the assumption that the opponent is at least as rational as we ourselves are and does everything to prevent us from achieving our object.

All recommendations in game theory are deduced from these principles. Consequently, no account is taken of risk elements, which are inevitably present in every real strategy, nor of possible miscalculations or errors made by the players.

Game theory, like any mathematical model of a complex phenomenon, has its limitations. The most serious limitation is the fact that the gain is artificially reduced to only one number. In most practical conflict situations, when working out a rational strategy one has to take into account several criteria of successful action, i. e. several numerical parameters rather than one. A strategy optimal according to one criterion is not necessarily optimal according to another. However, by realizing these limitations and therefore not blindly following the recommendations obtained by game theoretic methods, one can still employ mathematical game theory techniques to work out a strategy which would at any rate be "acceptable", if not "optimal".

2. The Lower and the Upper Value of the Game. The "Minimax" Principle

Consider an $m \times n$ game having the following matrix.

We shall denote by the letter i the number of our strategy and by the letter j the number of our opponent's strategy.

B / A	B_1	B_2		B_n
A_1	a_{11}	a_{12}		a_{1n}
A_2	a_{21}	a_{22}		a_{2n}
A_m	a_{m1}	a_{m2}	...	a_{mn}

We undertake to determine our optimal strategy. We shall analyse each of our strategies sequentially starting with A_1. Choosing strategy A_i we must always count on our opponent responding to it with that strategy B_j for which our gain a_{ij} is minimal. We shall determine this value of the gain, i. e. the smallest of the numbers a_{ij} in the ith row. We shall denote it by α_i:

$$\alpha_i = \min_j a_{ij} \tag{2.1}$$

Here the symbol $\min_j$ (minimum over j) denotes the minimum value of the given parameter for all possible j.

We shall write the numbers α_i next to the matrix above in an additional column.

By choosing some strategy A_i we can count on winning, as a result of our opponent's rational actions, not more than α_i. It is natural that, by acting in the most cautious way and assuming the most rational opponent (i. e. avoiding any risk), we must decide on a strategy A_i for which number α_i is maximal. We shall denote this maximal value by α:

$$\alpha = \max_i \alpha_i$$

or, taking into account formula (2.1)

$$\alpha = \max_i \min_j \alpha_{ij}$$

The quantity α is called the *lower value of the game*, or else the *maximin gain* or simply the *maximin*.

A \ B	B_1	B_2		B_n	α_i
A_1	a_{11}	a_{12}	...	a_{1n}	α_1
A_2	a_{21}	a_{22}		a_{2n}	α_2
A_m	a_{m1}	a_{m2}		a_{mn}	α_m
β_j	β_1	β_2		β_m	

The number α lies in some row of the matrix; player A's strategy which corresponds to this row is called a *maximin strategy.*

Evidently, if we keep to a maximin strategy, then, whatever our opponent's behaviour, we are *assured a gain which is in any case not less than* α. Because of this the quantity α is called the "lower value of the game". This is the guaranteed minimum which we can assure ourselves by keeping to the most cautious ("play safe") strategy.

Evidently, a similar argument can be carried out for opponent B. Since the opponent is interested in minimizing our gain, he must examine each of his strategies in terms of the maximum gain for it. Therefore we shall write out the maximal values a_{ij} for each column at the bottom of the matrix:

$$\beta_j = \max_i a_{ij}$$

and find the minimal value β_j:

$$\beta = \min_j \beta_j$$

or

$$\beta = \min_j \max_i a_{ij}$$

The quantity β is called the *upper value of the game*, or else the "minimax". The opponent's strategy corresponding to the minimax gain is called his "minimax strategy".

Keeping to his most cautious minimax strategy the opponent assures himself the following: whatever move we make against him, *he will lose a sum not greater than* β.

In game theory and its applications, the principle of cautiousness which dictates to the players the choice of the corresponding strategies (the maximin and the minimax one) is often called the "minimax principle". The most cautious maximin and minimax strategies for the players are designated by the general term "minimax strategies".

By way of illustration we shall determine the lower and the upper value of the game and the minimax strategies for the games in Examples 1, 2, and 3 of Section 1.

EXAMPLE 1. Example 1 of Section 1 gives a game with the matrix given below.

$A \backslash B$	B_1	B_2	α_i
A_1	1	−1	−1
A_2	−1	1	−1
β_j	1	1	

Since the values α_i and β_j are constant and are -1 and $+1$ respectively, the lower and upper values of the game are also -1 and $+1$:

$$\alpha = -1; \quad \beta = +1$$

Either strategy of player A is a maximin strategy, and either strategy of player B is a minimax strategy. The conclusion that follows is trivial: by adopting any of his strategies, player A can assure that his loss is not greater than 1; the same can also be guaranteed by player B.

EXAMPLE 2. Example 2 of Section 1 gives a game with the following matrix.

The lower value of the game $\alpha = -3$; the upper value of the game $\beta = 4$. Our maximin strategy is A_1; employing it systematically we can count on a gain not less than -3 (a loss not greater than 3). The opponent's minimax strategy is either of the

$A \backslash B$	B_1	B_2	B_3	α_i
A_1	2	−3	4	**−3**
A_2	−3	4	−5	−5
A_3	4	−5	6	−5
β_j	**4**	**4**	6	

strategies B_1 or B_2; employing them systematically he can in any case guarantee that his loss is not greater than 4. If we deviate from our maximin strategy (choose strategy A_2, for instance) our opponent may "punish" us for this by playing strategy B_3 and reducing our gain to -5; similarly our opponent's deviation from his minimax strategy may increase his loss to 6.

EXAMPLE 3. Example 3 of Section 1 gives a game with the matrix

$A \backslash B$	B_1	B_2	B_3	α_i
A_1	0.9	0.4	0.2	0.2
A_2	0.3	0.6	0.8	**0.3**
A_3	0.5	0.7	0.2	0.2
β_j	0.9	**0.7**	0.8	

The lower value of the game $\alpha = 0.3$; the upper value of the game $\beta = 0.7$. Our most cautious (maximin) strategy is A_2; using weapon A_2, we guarantee that we hit the aircraft on average at least 0.3 of the time. The most cautious (minimax) strategy for our opponent is B_2; using this aircraft he can be sure that it will be hit at most 0.7 of the time.

The last example can be very conveniently used to demonstrate one important property of minimax strategies, their instability. Suppose we are playing our most cautious (maximin) strategy A_2, while our opponent is playing his most cautious (minimax) strategy B_2. As long as both opponents are keeping to these strategies, the average gain is 0.6; it is greater than the lower value of the game but less than its upper value. Suppose now that our opponent learns that we are playing strategy A_2; he will immediately respond to it with strategy B_1 and reduce our gain to 0.3. We in turn have a good answer to strategy B_1, *namely strategy* A_1 which gives us a gain of 0.9, etc.

Thus the situation in which both players employ their minimax strategies is unstable and may be disturbed by information received about the opposite party's strategy.

There are some games, however, for which minimax strategies are stable. These are games for which the lower value is equal to the upper one:

$$\alpha = \beta$$

If the lower value of the game is equal to the upper one, then their common value is called the *net value of the game* (or simply the *value of the game*); we shall denote it by the letter ν.

Consider an example. Let a 4×4 game be given by the following matrix:

A \ B	B_1	B_2	B_3	B_4	α_i
A_1	0.4	0.5	0.9	0.3	0.3
A_2	0.8	0.4	0.3	0.7	0.3
A_3	0.7	**0.6**	0.8	0.9	**0.6**
A_4	0.7	0.2	0.4	0.6	0.2
β_j	0.8	**0.6**	0.8	0.9	

We find the lower value of the game:

$$\alpha = 0.6$$

We find the upper value of the game:

$$\beta = 0.6$$

They turn out to be the same; consequently, the game has a new value equal to $\alpha = \beta = \nu = 0.6$.

The entry 0.6 marked out in the game matrix is simultaneously *minimal in its row and maximal in its column*. In geometry a point on a surface with a similar property (being a minimum for one coordinate and simultaneously a maximum for the other) is called a *saddle point*; by analogy, the term is used in game theory as well. A matrix element possessing this property is called a *saddle point of the matrix*, and the game is said *to possess a saddle point*.

Corresponding to a saddle point are a pair of minimax strategies (A_3 and B_2 in this example). These are called *optimal* and their combination is called a *solution of the game*.

The solution of the game has the following remarkable property. If one of the players (for instance, A) keeps to his optimal strategy and the other (B) deviates in some way from his optimal strategy, then *there is never any advantage to the player* (B) *who has made the deviation*; such a deviation on the part of player B would at best leave the payoff unchanged or at worst increase it.

On the other hand, if B keeps to his optimal strategy and A deviates from his, then there is no possible advantage to A.

This statement is easy to verify using the example of the game with a saddle point that we have been considering.

We see that in the case of a game with a saddle point minimax strategies possess a peculiar "stability": if one party adheres to its minimax strategy, then it can be only disadvantageous for the other to deviate from his. Notice that in this case neither player's knowledge that his opponent has chosen his optimal strategy could change his own behaviour: if he does not want to act against his own interests, he must adhere to his optimal strategy. A pair of optimal strategies in a game with a saddle point are a kind of "equilibrium position": any deviation from his optimal strategy leads the deviating player to disadvantageous consequences forcing him to return to his initial position.

So for every game with a saddle there is a solution giving a pair of optimal strategies of both parties which possesses the following properties.

(1) If both parties adhere to their optimal strategies, then the average gain is equal to the net value of the game, v, which is at the same time its lower and upper value.

(2) If one of the parties adheres to its optimal strategy and the other deviates from its, then this would only make the deviating party lose and in no case would increase its gain.

The class of games having a saddle point is of great interest from both theoretical and practical points of view.

Game theory proves that every game with perfect information has a saddle point and that consequently every game of this kind has a solution, i. e. there is a pair of optimal strategies, one for each player which gives an average gain equal to the value of the game. If a perfect-information game contains only personal moves, then, if each player plays his optimal strategy, the game must always end with a definite outcome, namely a gain exactly equal to the value of the game.

As an example of a game with perfect information we mention the well-known game of laying coins on a round table. Two players put in turn similar coins on a round table, choosing each time an arbitrary position for the centre of the coin; no mutual covering of the coins is allowed. The victor is the player who puts down the last coin (when there is no place to put down any others). It is obvious that the outcome of this game is always predetermined, and there is a quite definite strategy guaranteeing a sure win for the player who is the first to put down a coin. That is, he must put the first coin in the centre of the table and then make a symmetrical move in answer to each move of his opponent. Here the second player may behave in any way he likes without changing the predetermined outcome of the game. Therefore this game makes sense only to players who do not know the optimal strategy. The situation is the same in chess and other games with perfect information; all these games have a saddle point and a solution which shows each of the players his optimal strategy; no solution of the game of chess has been found, only because the number of combinations of possible moves in chess is too large for a game matrix to be constructed and a saddle point to be found in it.

3. Pure and Mixed Strategies. The Solution of a Game in Mixed Strategies

Games with a saddle point are rarely met with among finite games of practical importance; more typical is the case where the lower and the upper value of a game are different. By analysing

the matrices of such games we have come to the conclusion that, if either player is allowed a choice of only one strategy, then counting on a rationally acting opponent this choice must be determined by the minimax principle. Keeping to our maximin strategy we clearly guarantee ourselves a gain equal to the lower value of the game, α, whatever our opponent's behaviour may be. The natural question arises: Is it possible to guarantee oneself an average gain greater than α if one alternates randomly several strategies rather than playing only one "pure" strategy?

In game theory such combined strategies consisting of several pure strategies which are alternated randomly with a definite ratio of frequencies are called *mixed strategies.*

It is obvious that every pure strategy is a particular case of mixed strategy in which all strategies but one are played with zero frequencies and the given strategy is played with a frequency of 1.

It turns out that by playing not only pure but also mixed strategies one can find a solution for every finite game, i. e. a pair of strategies (mixed strategies in the general case) such that, if they are used by both players, the gain will be equal to the value of the game, and such that if the optimal strategy is deviated from unilaterally by any of the players, the gain may change only in the direction disadvantageous to the deviating player.

This statement constitutes the substance of the so-called *fundamental theorem of game theory* which was first proved in 1928 by von Neumann. The existing proofs of the theorem are rather complicated; therefore we shall only state it.

Every finite game has at least one solution (possibly in the range of mixed strategies).

The gain derived from a solution is called the value of the game. It follows from the fundamental theorem that every finite game has a value. It is obvious that the value of a game, ν, is always between the lower value of the game α and the upper value of the game β:

$$\alpha \leqslant \nu \leqslant \beta \tag{3.1}$$

Indeed, α is the maximum safe gain that we can assure ourselves playing only our pure strategies. Since mixed strategies include all the pure strategies as particular cases, by admitting mixed strategies as well as the pure strategies, we at any rate do not make our possibilities worse; consequently,

$$\nu \geqslant \alpha$$

Similarly, considering our opponent's possibilities, we show that

$$\nu \leqslant \beta$$

Inequality (3.1) follows from these.

We introduce a special notation for mixed strategies. If, for example, our mixed strategy consists in playing strategies A_1, A_2, and A_3 with frequencies p_1, p_2, and p_3, with $p_1 + p_2 + p_3 = 1$, we shall denote it by

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ p_1 & p_2 & p_3 \end{pmatrix}$$

Similarly, our opponent's mixed strategy will be denoted by

$$S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ q_1 & q_2 & q_3 \end{pmatrix}$$

where q_1, q_2, and q_3 are the frequencies in which strategies B_1, B_2, and B_3 are mixed; $q_1 + q_2 + q_3 = 1$.

Suppose that we have found a solution of the game which consists of two optimum mixed strategies, S_A^* and S_B^*. In the general case, not all pure strategies available to a given player enter into his mixed strategy. We shall call the strategies entering into a player's optimum mixed strategy his "utility" strategies.

It turns out that the solution of a game possesses still another remarkable property: *if one of the players keeps to his optimum mixed strategy S_A^* (S_B^*), then the gain remains unchanged and equal to the value of the game, ν, no matter what the other player does, provided he keeps within his "utility" strategies.* For example, he may play any of his "utility" strategies in pure form or mix them in any proportions.

Let us prove this statement. Suppose there is a solution S_A^*, S_B^* of an $m \times n$ game. For the sake of concreteness, we shall consider that the optimum mixed strategy S_A^* consists of a mixture of three "utility" strategies, A_1, A_2, and A_3; correspondingly, S_B^* consists of a mixture of three "utility" strategies B_1, B_2, B_3:

$$S_A^* = \begin{pmatrix} A_1 & A_2 & A_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \qquad S_B^* = \begin{pmatrix} B_1 & B_2 & B_3 \\ q_1 & q_2 & q_3 \end{pmatrix}$$

with $p_1 + p_2 + p_3 = 1$; $q_1 + q_2 + q_3 = 1$. It is asserted that if we adhere to strategy S_A^*, then our opponent may play strategies B_1, B_2, B_3 in any proportions and the gain will remain unchanged and be as before equal to the value of the game, ν.

We prove this as follows. Let ν_1, ν_2, ν_3 be the gains when we play our strategy S_A^* and our opponent plays his strategies B_1, B_2, and B_3, respectively.

It follows from the definition of an optimal strategy that no deviation of our opponent from this strategy S_B^* can be advantageous to him, therefore

$$\nu_1 \geqslant \nu; \quad \nu_2 \geqslant \nu; \quad \nu_3 \geqslant \nu$$

We shall see whether the value of ν_1, ν_2, or ν_3 can, at least in one of the three cases, be found to be *greater* than ν. It turns out that it cannot. Indeed, let us express the gain ν for the optimal strategies S_A^*, S_B^* in terms of the gains ν_1, ν_2, ν_3. Since in S_B^* strategies B_1, B_2, and B_3 are played with frequencies q_1, q_2, q_3, we have

$$\nu = \nu_1 q_1 + \nu_2 q_2 + \nu_3 q_3 \tag{3.2}$$

$$(q_1 + q_2 + q_3 = 1)$$

It is obvious that, if at least one of the quantities ν_1, ν_2, ν_3 were greater than ν, then their mean weighted value (3.2) would also be greater than ν, which contradicts the condition. Thus a very important property of optimal strategies is proved; it will be extensively used in solving games.

4. Elementary Methods for Solving Games. 2×2 and $2 \times n$ Games

If an $m \times n$ game has no saddle point, then finding a solution is in general a very difficult problem, especially with large m and n.

Sometimes one can simplify this problem if one can reduce the number of strategies by deleting superfluous ones.

Strategies are superfluous if they are (a) duplicated or (b) clearly disadvantageous. Consider, for example, a game with the following matrix:

A \ B	B_1	B_2	B_3	B_4
A_1	1	2	4	3
A_2	0	2	3	2
A_3	1	2	4	3
A_4	4	3	1	0

It can easily be seen that strategy A_3 precisely repeats ("duplicates") strategy A_1, therefore any of these two strategies can be deleted.

Further, by comparing the elements of rows A_1 and A_2 we see that each entry in row A_2 is less than (or equal to) the corresponding entry in row A_1. It is obvious that we should never use strategy A_2; for it is clearly disadvantageous. Deleting A_3 and A_2 reduces the matrix to a simpler form (see below).

$A \backslash B$	B_1	B_2	B_3	B_4
A_1	1	2	4	3
A_4	4	3	1	0

Further notice that strategy B_3 is clearly disadvantageous to our opponent; deleting it reduces the matrix to its final form (see below). Thus, by deleting the duplicated and clearly disadvantageous strategies, the 4×4 game is reduced to a 2×3 game.

$A \backslash B$	B_1	B_2	B_4
A_1	1	2	3
A_4	4	3	0

The procedure of deleting duplicated and clearly disadvantageous strategies should always be carried out before solving the game.

The simplest cases of the finite games that can always be solved by elementary methods are 2×2 and $2 \times m$ games.

Consider a 2×2 game with the following matrix. Two cases can occur here: (1) the game has a saddle point; (2) the game has no saddle point. In the first case the solution is obvious, it is a pair of strategies intersecting in the saddle point. Notice by the way that the presence of a saddle point in a 2×2 game

A \ B	B_1	B_2
A_1	a_{11}	a_{12}
A_2	a_{21}	a_{22}

always corresponds to the existence of clearly disadvantageous strategies. These should be deleted at the outset *.

Suppose there is no saddle point and, consequently, the lower value of the game is not equal to the upper value $\alpha \neq \beta$. Find an optimum mixed strategy for player A:

$$S_A^* = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix}$$

It is notable for the property that whatever our opponent does (provided he does not deviate from his "utility" strategies) the gain (payoff) will be equal to the value of the game, v. In a 2×2 game both strategies of our opponent are "utility" strategies, otherwise the game would have a solution in the range of pure strategies (there would be a saddle point). It means that if we adhere to our optimal strategy $S^* = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix}$, then our opponent may play any of his pure strategies B_1, B_2 without changing the average payoff v. Thus we have two equations:

$$\left.\begin{aligned} a_{11}p_1 + a_{21}p_2 &= \nu \\ a_{21}p_1 + a_{22}p_2 &= \nu \end{aligned}\right\} \qquad (4.1)$$

from which, taking into account the fact that $p_1 + p_2 = 1$, we get

$$a_{11}p_1 + a_{21}(1 - p_1) = a_{12}p_1 + a_{22}(1 - p_1)$$

$$p_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}} \qquad (4.2)$$

The value of the game v is found by substituting the values of p_1, p_2 into any of the equations (4.1).

* It is left for the reader to check this for himself with a number of matrices.

If the value of the game is known, then one equation is enough to determine our opponent's optimal strategy $S_B^* = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}$, for example:

$$a_{11}q_1 + a_{12}q_2 = \nu$$

from which, considering that $q_1 + q_2 = 1$, we have

$$q_1 = \frac{\nu - a_{12}}{a_{11} - a_{12}} \quad q_2 = 1 - q_1$$

EXAMPLE 1. We find the solution of the 2×2 game considered in Example 1 of Section 1 having the following matrix.

A \ B	B_1	B_2
A_1	1	-1
A_2	-1	1

The game has no saddle point ($\alpha = -1$; $\beta = +1$) and, consequently, the solution must lie in the range of mixed strategies:

$$S_A^* = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix} \qquad S_B^* = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}$$

It is necessary to find p_1, p_2, q_1, and q_2.

For p_1 we have the equation

$$1 \times p_1 + (-1) \times (1 - p_1) = (-1) \times p_1 + 1 \times (1 - p_1)$$

from which

$$p_1 = {}^1/_2 \quad p_2 = {}^1/_2$$

Similarly we find

$$q_1 = {}^1/_2 \quad q_2 = {}^1/_2 \quad \nu = 0$$

Consequently, the optimal strategy for either player is to alternate his pure strategies at random, employing each as often as the other; here the average payoff will be zero.

The conclusion we have come to was sufficiently clear beforehand. In the next example we shall consider a more complex game whose

solution is not so obvious. The example is an elementary game of the sort known as "bluffing" games. It is common practice in conflict situations to use various methods of bluffing, or misleading one's opponent (misinformation, arrangement of dummy targets, etc.). The example is quite instructive in spite of its simplicity.

EXAMPLE 2. The game is as follows. There are two cards, an ace and a deuce. Player A draws either of the two at random; B does not see which card is drawn. If A has drawn the ace, he says "I've got the ace" and demands a dollar from his opponent. If A has drawn the deuce, then he may either (A_1) say "I've got the ace" and demand a dollar from his opponent or (A_2) confess that he has got the deuce and pay his opponent a dollar.

The opponent, if he is paid the dollar voluntarily, can only accept it. If, however, a dollar is demanded from him, then he may either (B_1) believe that player A has got the ace and give him the dollar or (B_2) demand a check so as to see whether A's statement is true or not. If it is found that A does have the ace, B must pay A two dollars. If, however, it is found that A is bluffing B and has the deuce, player A pays B two dollars.

Analyse the game and find the optimal strategy for each player.

SOLUTION. The game has a comparatively complex structure; it consists of one obligatory chance move – player A's choice of one of the two cards – and two personal moves which may or may not be realized, however. Indeed, if A has drawn the ace, he makes no personal move: he is given only one possibility, that of demanding a dollar, which he does. In this case the personal move – to believe or not to believe (i. e. to pay or not to pay the dollar) – is passed to player B. If A has got the deuce as a result of his first chance move, then he is given a personal move: to pay a dollar or try to bluff his opponent and demand a dollar (in short, "not to bluff" or "to bluff"). If A decides on the first choice, then it remains for B to accept the dollar; if A decides on the second choice, then player B is given a personal move: to believe or not to believe A (i. e. to pay A a dollar or demand a check).

The strategies of each of the players are the rules indicating how the player should act when given a personal move.

Evidently A has only two strategies:

to bluff, A_1 and not to bluff, A_2.

B has also two strategies:

to believe, B_1, and not to believe, B_2.

We construct the matrix of the game. To this end we calculate the average payoff for each combination of strategies.

1. A_1B_1 (A is bluffing, B believes).

If A has got the ace (the probability of which is $^1/_2$), then he is not given any personal move; he demands a dollar and player B believes him; the payoff is 1.

If A has got the deuce (the probability of which is also $^1/_2$), then in accordance with his strategy he bluffs and demands a dollar; B believes him and pays him the money; A's payoff is also 1.

The average payoff is

$$a_{11} = {}^1/_2 \times 1 + {}^1/_2 \times 1 = 1$$

2. A_1B_2 (A is bluffing, B does not believe).

If A has got the ace, he has no personal move; he demands a dollar; in accordance with his strategy, B does not believe A and as a result of the check pays A two dollars (A's payoff is $+2$).

If A has got the deuce, then in accordance with his strategy he demands a dollar; B, in accordance with his, does not believe him; as a result A pays him two dollars (A's payoff is -2). The average payoff is

$$a_{12} = {}^1/_2 \times (+2) + {}^1/_2 \times (-2) = 0$$

3. A_2B_1 (A is not bluffing, B believes).

If A has drawn the ace, he demands a dollar; in accordance with his strategy, B pays the money; A's payoff is 1. If A has drawn the deuce, then in accordance with his strategy he pays B a dollar; it remains for B only to take the money (A's payoff is -1). The average payoff is

$$a_{21} = {}^1/_2 \times (+1) + {}^1/_2 \times (-1) = 0$$

4. A_2B_2 (A is not bluffing, B does not believe).

If A has drawn the ace, he demands a dollar; B checks and as a result pays A two dollars (the payoff is $+2$).

If A has drawn the deuce, he pays B a dollar; it only remains for B to take the money (the payoff is -1).

The average payoff is

$$a_{22} = {}^1/_2 \times (+2) + {}^1/_2 \times (-1) = {}^1/_2$$

We construct the matrix of the game (see below). It has no saddle point. The lower value of the game is $\alpha = 0$; the upper value of the game is $\beta = {}^1/_2$. We find the solution of the game in the range

B / A	B_1 (to believe)	B_2 (not to believe)
A_1 (to bluff)	1	0
A_2 (not to bluff)	0	$^1/_2$

of mixed strategies. Applying formula (4.2), we have

$$p_1 = \frac{^1/_2}{1 + {}^1/_2} = {}^1/_3 \quad p_2 = {}^2/_3 \quad S_A^* = \begin{pmatrix} A_1 & A_2 \\ ^1/_3 & ^2/_3 \end{pmatrix}$$

i. e. player A should play his first strategy (which is to bluff) one-third of the time and his second strategy (not to bluff) two-thirds of the time. Then he will gain on average the value of the game

$$\nu = {}^1/_3$$

The value $\nu = {}^1/_3$ implies that under these circumstances the game is advantageous to A and disadvantageous to B. Playing his optimal strategy, A can always assure himself a positive average payoff.

Notice that if A played his most cautious (maximin) strategy (both strategies, A_1 and A_2, are maximin in this case), he would have an average payoff equal to zero. Thus a mixed strategy enables A to realize his advantage over B arising from the given rules of the game.

We determine the optimal strategy for B. We have

$$q_1 \times 1 + q_2 \times 0 = {}^1/_3 \quad q_1 = {}^1/_3 \quad q_2 = {}^2/_3$$

whence $S_B^* = \begin{pmatrix} B_1 & B_2 \\ ^1/_3 & ^2/_3 \end{pmatrix}$, i. e. player B should believe A and pay him a dollar without a check one-third of the time and should check two-thirds of the time. Then he will on average lose $^1/_3$ of a dollar for each game. If he played his pure minimax strategy B_2 (not to believe), then he would on average lose $^1/_2$ of a dollar for each game.

The solution of a 2×2 game can be given a simple geometrical interpretation. Let there be a 2×2 game with the matrix below.

Take a section of the axis of abscissae of length 1 (Fig. 4.1). The left-hand end of the section (the point with the abscissa $x = 0$) will

B \ A	B_1	B_2
A_1	a_{11}	a_{12}
A_2	a_{21}	a_{22}

represent strategy A_1; the right-hand end of the section ($x = 1$) will represent strategy A_2. Draw through the point A_1 and A_2 two perpendiculars to the axis of abscissae; the axis *I-I* and the axis *II-II*. We shall use the axis *I-I* to plot payoffs for strategy

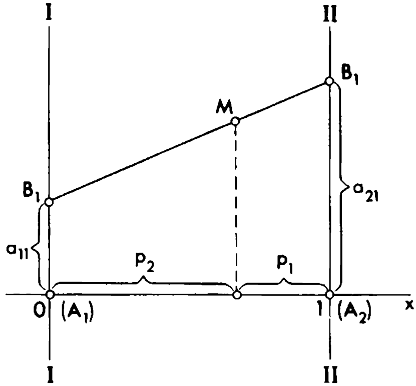

Fig. 4.1

A_1 and the axis *II-II* to plot payoffs for strategy A_2. Consider our opponent's strategy B_1, it gives two points on the axes *I-I* and *II-II* with the ordinates a_{11} and a_{21}, respectively. Draw through these points the line B_1B_1. It is obvious that if we play the mixed strategy $S_A^* = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix}$ against our opponent's strategy B_1 then our average payoff, equal to $a_{11}p_1 + a_{12}p_2$ in this case, will be represented by the point M on the line B_1B_1, the abscissa of the point is equal to p_2. We shall adopt the convention to refer to the line B_1B_1 representing the payoff when strategy B_1 is played as "strategy B_1".

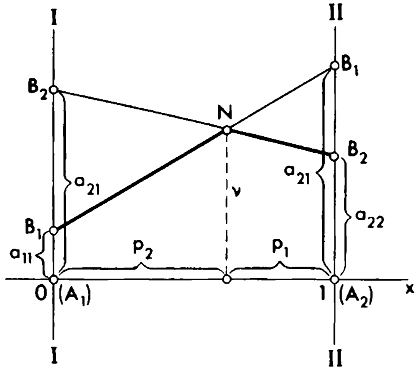

Fig. 4.2

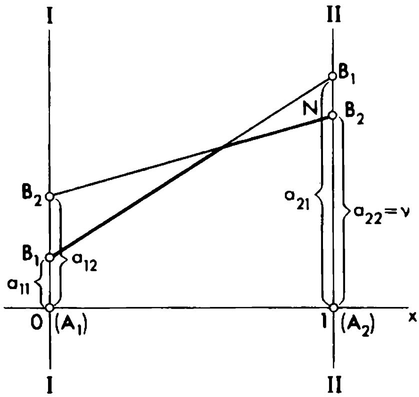

Fig. 4.3

Evidently, in precisely the same way it is possible to construct strategy B_2 as well (Fig. 4.2).

We must find an optimal strategy S_A^*, i. e. such that the minimum payoff for it (whatever B's behaviour) would be maximized. To this end, construct the *lower bound for the payoff* for strategies B_1, B_2, i. e. the broken line B_1NB_2 marked in bold type in Fig. 4.2. This lower bound will express the minimum payoff for any of player A's mixed strategies; it is the point N at which this minimum payoff attains the maximum that determines the solution and the value of the game. It is easy to make sure that the ordinate of the point N

is the value of the game, v, and that its abscissa is equal to p_2, the relative frequency with which strategy A_2 is played in the optimum mixed strategy S_A^*.

In our case the solution of the game was given by the point of intersection of the strategies. This is not always the case, however; Fig. 4.3 shows a case where in spite of the presence of an intersection of the strategies the solution gives pure strategies (A_2 and B_2) for both players, and the value of the game $v = a_{22}$.

In this case the matrix has a saddle point, and strategy A_1 is clearly disadvantageous, since, whatever our opponent's pure strategy, it gives a smaller payoff than A_2.

In the case where our opponent has a clearly disadvantageous strategy, the geometrical interpretation is of the form shown in Fig. 4.4.

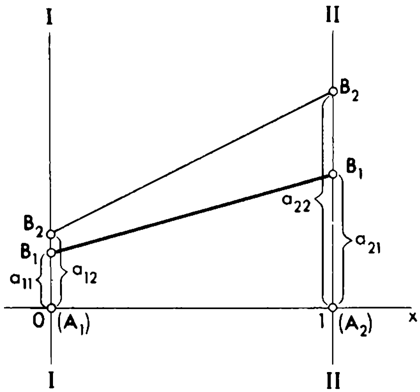

Fig. 4.4

In this case the lower bound of the payoff coincides with strategy B_1; strategy B_2 is clearly disadvantageous to our opponent.

Geometrical interpretation makes it possible to represent graphically the lower and the upper value of the game as well (Fig. 4.5). By way of illustration we shall construct geometrical interpretations for the 2×2 games of Examples 1 and 2 (Figs. 4.6 and 4.7).

We have made sure that any 2×2 game can be solved by elementary methods. It is possible to solve in a quite similar way any $2 \times n$ game where we have only two strategies and our opponent has an arbitrary number of strategies.

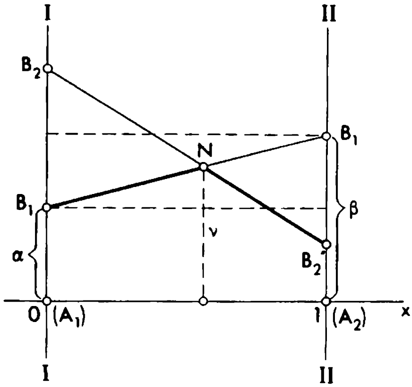

Fig. 4.5

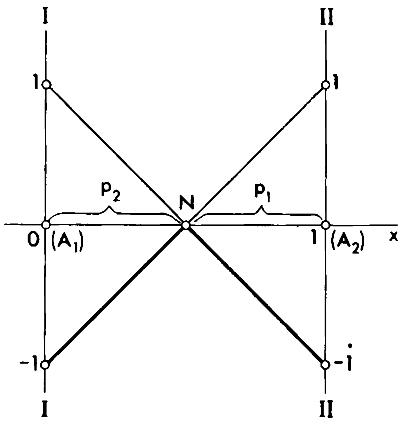

Fig. 4.6

Suppose we have two strategies A_1, A_2 at our disposal and our opponent has n strategies: B_1, B_2, ..., B_n. The matrix $\|a_{ij}\|$ is given; it consists of two rows and n columns. As in the case of two strategies we give a geometrical interpretation of the problem; our opponent's n strategies will be represented by n straight lines (Fig. 4.8). We construct the lower bound for the payoff (the broken line B_1MNB_2) and find in it the point N with the maximum ordinate.

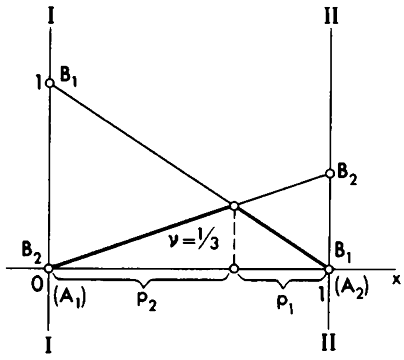

Fig. 4.7

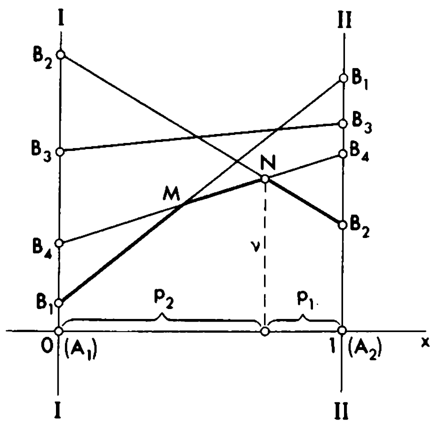

Fig. 4.8

This point gives a solution to the game $\left(\text{strategy } S_A^* = \right.$ $\left. = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix}\right)$; the ordinate of the point N is equal to the value of the game, ν, and the abscissa is equal to the relative frequency p_2 of strategy A_2.

In the case at hand our opponent's optimal strategy results from playing a mixture of two "utility" strategies, B_2 and B_4, intersecting in

the point N. Strategy B_3 is clearly disadvantageous and strategy B_1 is disadvantageous when the optimal strategy S_A^* is played. If A adheres to his optimal strategy, then the payoff will remain unchanged no matter which of his "utility" strategies B may play; it will change, however, if B passes to his strategies B_1 or B_3.

It is proved in game theory that every $m \times n$ game has a solution in which the number of "utility" strategies for either party does not exceed the smaller of the two numbers m and n. From this it follows, in particular, that a $2 \times m$ game has always a solution in which there are at most two "utility" strategies for either party.

By making use of a geometrical interpretation it is possible to give a simple method for solving any $2 \times m$ game. From the graph we find a pair of our opponent's "utility" strategies, B_j and B_k, intersecting in the point N (if there are more than two strategies intersecting in the point N, we take any two of them). We know that if player A keeps to his optimal strategy, then the payoff does not depend on the proportion in which B plays his "utility" strategies; consequently

$$\left.\begin{aligned} p_1 a_{1j} + p_2 a_{2j} = \nu \\ p_1 a_{1k} + p_2 a_{2k} = \nu \end{aligned}\right\}$$

From these equations and the condition $p_2 = 1 - p_1$ we find p_1, p_2 and the value of the game, ν.

Knowing the value of the game, one can at once determine player B's optimal strategy $S_B^* = \begin{pmatrix} B_j & B_k \\ q_j & q_k \end{pmatrix}$.

To do this, one solves, for example, the equation

$$q_j q_{1j} + q_k a_{1k} = \nu$$

where

$$q_j + q_k = 1$$

In the case where we have m strategies at our disposal while our opponent has only two, the problem is obviously solved in a quite similar way; suffice it to say that by changing the sign of the payoff it is possible to turn player A from the "winner" into the "loser". It is also possible to solve the game without changing the sign of the payoff, then the problem is solved for B directly, but the upper bound of the payoff is constructed rather than the lower bound (Fig. 4.9). One looks for the point N on the bound with the minimum ordinate, which is precisely the value of the game, ν.

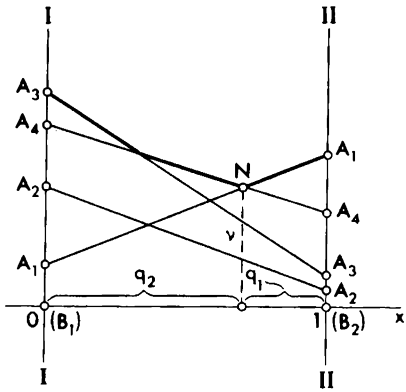

Fig. 4.9

We shall consider and solve several examples of 2×2 and $2 \times m$ games which are simplified cases of games of practical importance.

EXAMPLE 3. Side A is sending two bombers, I and II, to the enemy's assembly area; I is flying in front, and II is flying behind. One of the bombers, which precisely it is not known beforehand, is carrying a bomb, the other is acting as an escort.

Over the enemy's area the bombers are attacked by side B's fighter. The bombers are armed with guns having different rates of fire. If the fighter attacks the back bomber II, then it is fired at only by this bomber's guns; if, however, it attacks the front bomber, then it is fired at by both bombers. The fighter is hit with a probability of 0.3 in the first case and that of 0.7 in the second.

If the fighter is not shot down by the defensive fire of the bombers, then it hits the target it chooses with a probability of 0.6. The task of the bombers is to carry the bomb to the target; the task of the fighter is to prevent this, i. e. to shoot down the bomber carrying the bomb. Choose optimal strategies for each side:

(a) for side A: which bomber is to carry the bomb?

(b) for side B: which bomber is to be attacked?

SOLUTION. We have here a simple case of a 2×2 game; the payoff is the probability of the bomber carrying the bomb remaining unhit.

Our strategies are:

A_1: bomber I is to carry the bomb;

A_2: bomber *II* is to carry the bomb.

The enemy's strategies are:

B_1: bomber *I* is to be attacked;

B_2: bomber *II* is to be attacked.

We set up the matrix of the game, i. e. find the average payoff for each combination of strategies.

1. A_1B_1 (*I* is carrying a bomb, *I* is attacked).

The bomber carrying a bomb will not be hit, if the fighter is shot down by the bombers or if it is not shot down but misses its target:

$$a_{11} = 0.7 + 0.3 \times 0.4 = 0.82$$

2. A_2B_1 (*II* is carrying the bomb, *I* is attacked).

$$a_{21} = 1$$

3. A_1B_2 (*I* is carrying the bomb, *II* is attacked)

$$a_{12} = 1$$

4. A_2B_2 (*II* is carrying the bomb, *II* is attacked)

$$a_{22} = 0.3 + 0.7 \times 0.4 = 0.58$$

The matrix of the game is of the form

$A \backslash B$	B_1	B_2
A_1	0.82	1
A_2	1	0.58

The lower value of the game is 0.82; the upper value is *1*. The matrix of the game has no saddle point; we seek the solution in the range of mixed strategies.

We have

$$p_1 \times 0.82 + p_2 \times 1 = v$$

$$p_1 \times 1 + p_2 \times 0.58 = v$$

$$p_2 = 1 - p_1$$

Those equations yield

$$p_1 = 0.7 \quad p_2 = 0.3$$

Our optimal strategy is

$$S_A^* = \begin{pmatrix} A_1 & A_2 \\ 0.7 & 0.3 \end{pmatrix}$$

i. e. bomber I should be chosen to carry a bomb more often than II. The value of the game is

$$\nu = 0.874$$

Knowing ν we determine q_1 and q_2, the relative frequencies for strategies B_1 and B_2 in the enemy's optimal strategy S_B^*. We have

$$q_1 \times 0.82 + q_2 \times 1 = 0.874$$

$$q_2 = 1 - q_1$$

thence

$$q_1 = 0.7 \quad q_2 = 0.3$$

i. e. the enemy's optimal strategy is

$$S_B^* = \begin{pmatrix} B_1 & B_2 \\ 0.7 & 0.3 \end{pmatrix}$$

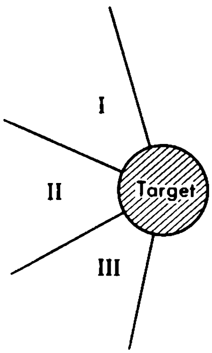

Fig. 4.10

EXAMPLE 4. Side A is attacking a target, side B is defending it. Side A has two aircraft; side B has three anti-aircraft guns. Each aircraft carries a powerful weapon; for the target to be hit, it is enough for at least one aircraft to break through to it. Side A's aircraft may choose any of the three directions, I, II, or III (Fig. 4.10).

The enemy (side B) may place any of its guns in any direction, each gun raking only the area of space covering one

direction and leaving unraked the other directions. Each gun may fire at one aircraft only; the aircraft fired at is hit with a probability of 1. Side A does not know where the guns are placed; side B does not know from where the aircraft will appear. Side A's task is to hit the target; side B's task is to prevent it from being hit. Find the solution of the game.

SOLUTION. This game is a 2×3 game. The payoff is the probability of hitting the target. Our possible strategies are:

A_1: to send one aircraft in one of the three directions and one in another;

A_2: to send both aircraft in one of the directions.

The enemy's strategies are:

B_1: to place a gun in each of the directions;

B_2: to place two guns in one of the directions and one in another;

B_3: to place all the three guns in one direction. We set up the matrix.

1. A_1B_1 (the aircraft are flying in different directions; the guns are placed one by one). Clearly no aircraft will break through to the target in this situation:

$$a_{11} = 0$$

2. A_2B_1 (the aircraft are flying together in one direction; the guns are placed one by one). In this situation, one aircraft will evidently pass to the target without being fired at:

$$a_{21} = 1$$

3. A_1B_2 (the aircraft are flying one by one, the enemy is protecting two directions, leaving a third unprotected). The probability of at least one aircraft breaking through to the target is just the probability of one of them choosing the unprotected direction:

$$a_{12} = {}^2/_3$$

4. A_2B_2 (the aircraft are flying together in one direction; the enemy is protecting one direction with two guns and another with one gun, i. e. it is protecting one direction, leaving the other two practically unprotected). The probability of at least one aircraft breaking through to the target is just the probability of the pair of aircraft choosing a practically unprotected direction:

$$a_{22} = {}^2/_3$$

5. A_1B_3 (the aircraft are flying separately, the enemy is protecting only one direction with the three guns)

$$a_{13} = 1$$

6. A_2B_3 (the aircraft are both flying together; the enemy is protecting only one direction with the three guns). For the target to be hit, the aircraft should both choose an unprotected direction:

$$a_{23} = {}^2/_3$$

The matrix of the game is as follows:

$A \backslash B$	B_1	B_2	B_3
A_1	0	$^2/_3$	1
A_2	1	$^2/_3$	$^2/_3$

It can be seen from the matrix that strategy B_3 is clearly disadvantageous in comparison with B_2 (this could have been determined beforehand). Deleting strategy B_3 reduces the game to a 2×2 one:

$A \backslash B$	B_1	B_2
A_1	0	$^2/_3$
A_2	1	$^2/_3$

The matrix has a saddle point; the lower value of the game (2/3) coincides with the upper one.

Notice simultaneously that strategy A_1 is clearly disadvantageous to us (A). The conclusion is that both sides, A and B, should always play their pure strategies A_2 and B_2, i. e. we should send the aircraft in twos, choosing randomly the direction in which to send the pair; the enemy should place its guns as follows: two in one direction, one in another; the choice of these directions should also be made at random (here "pure strategies" are seen to include an element of chance). Playing these optimal strategies, we shall always get an average payoff of $^2/_3$ (i. e. the target will be hit with a probability of $^2/_3$).

Notice that the solution of the game we have obtained is not unique; besides a pure strategy solution, there are a whole range of mixed solutions for player A that are optimal, from $p_1 = 0$ to $p_1 = {}^1/_3$ (Fig. 4.11). For example, it is easy to see directly that the same average payoff of ${}^2/_3$ will result if we play our optimal strategies A_1 and A_2 in the ratio of ${}^1/_3$ and ${}^2/_3$.

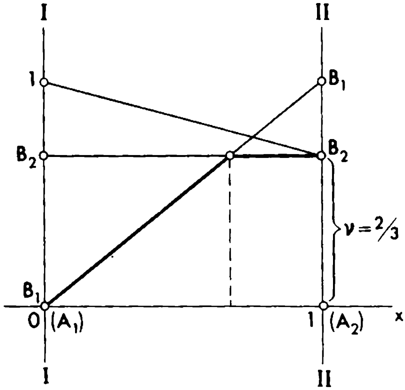

Fig. 4.11

EXAMPLE 5. The conditions are the same as in the preceding example, but we may strike a blow from four directions and the enemy has four guns at its disposal.

Solution. As before we have two possible strategies:

A_1: to send the aircraft one by one;

A_2: to send both aircraft together.

The enemy has five possible strategies:

$B_1(1 + 1 + 1 + 1)$: to place a gun in each direction;

$B_2(2 + 2)$: to place two guns in one direction and two in some other;

$B_3(2 + 1 + 1)$: to place two guns in one direction and one in each of some other two;

$B_4(3 + 1)$: to place three guns in one direction and one in some other;

$B_5(4)$: to place all the four guns in one direction.

Discard strategies B_4, B_5 beforehand as clearly disadvantageous. Arguing as in the previous example, construct the matrix of the game:

B / A	B_1 (1 + 1 + 1 + 1)	B_2 (2 + 2)	B_3 (2 + 1 + 1)
A_1	0	$^5/_6$	$^1/_2$
A_2	1	$^1/_2$	$^3/_4$

The lower value of the game is $^1/_2$, the upper one is $^3/_4$.

The matrix has no saddle point; the solution lies in the range of mixed strategies. Making use of the geometrical interpretation (Fig. 4.12), we single out the enemy's "utility" strategies: B_1 and B_2.

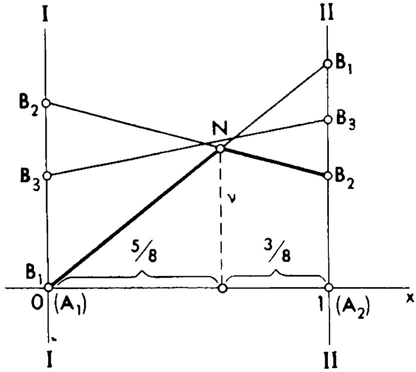

Fig. 4.12

The frequencies p_1 and p_2 are determined from the equations

$$p_1 \times 0 + (1 - p_1) \times 1 = v$$

$$p_1 \times {}^5/_6 + (1 - p_1) \times {}^1/_2 = v$$

thence

$$p_1 = {}^3/_8$$

$$p_2 = {}^5/_8$$

$$v = {}^5/_8$$

i. e. our optimal strategy is

$$S_A^* = \begin{pmatrix} A_1 & A_2 \\ ^3/_8 & ^5/_8 \end{pmatrix}$$

Playing it we assure ourselves an average payoff of $^5/_8$. Knowing the value of the game $v = 5/8$, we find the frequencies q_1 and q_2 for the enemy's "utility" strategies:

$$q_1 \times 0 + (1 - q_1) \times {}^5/_6 = {}^5/_8$$

$$q_1 = {}^1/_4 \quad q_2 = {}^3/_4$$

The optimal strategy for the enemy will be

$$S_B^* = \begin{pmatrix} B_1 & B_2 \\ ^1/_4 & ^3/_4 \end{pmatrix}$$

EXAMPLE 6. Side A has two strategies, A_1 and A_2, and side B has four, B_1, B_2, B_3 and B_4. The matrix of the game is of the form:

B / A	B_1	B_2	B_3	B_4
A_1	3	4	10	12
A_2	8	4	3	2

Find the solution of the game.

SOLUTION. The lower value of the game is 0.3; the upper value is 0.4. The geometrical interpretation (Fig. 4.13) shows that player B's utility strategies are B_1 and B_2 or B_2 and B_4. Player A has an infinite number of optimum mixed strategies: in an optimal strategy, p_1 may change from 1/5 to 4/5. The value of the game $v = 4$. The pure optimal strategy for player B is B_2.

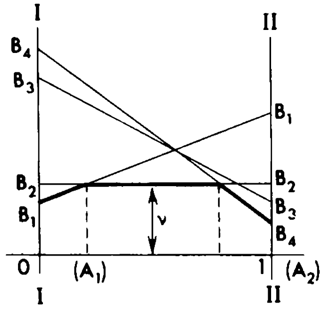

Fig. 4.13

5. General Methods for Solving Finite Games

So far we have considered only the most elementary games of the $2 \times n$ type which are very easy to solve and allow convenient and graphic geometrical interpretation.

In the general case, solving an $m \times n$ game is a rather difficult problem, as the complexity of the problem and the amount of calculation involved sharply increases with increasing m and n. These difficulties are not fundamental, however, and are associated only with the very large amount of calculation involved, which in some cases may prove impracticable. Essentially, the method of solution remains the same for any m.

To illustrate, consider an example of a $3 \times n$ games. We shall give it a geometrical interpretation, a three-dimensional one this time. Our three strategies, A_1, A_2 and A_3, will be represented by three points in the xOy plane; the first lies at the origin of coordinates (Fig. 5.1), the second and the third are in the Ox and Oy axes at distances of 1 from the origin.

Through the points A_1, A_2, and A_3, the axes *I-I*, *II-II*, and *III-III* are drawn perpendicular to the xOy plane. The axis *I-I* is used to plot the payoff for strategy A_1 and the axes

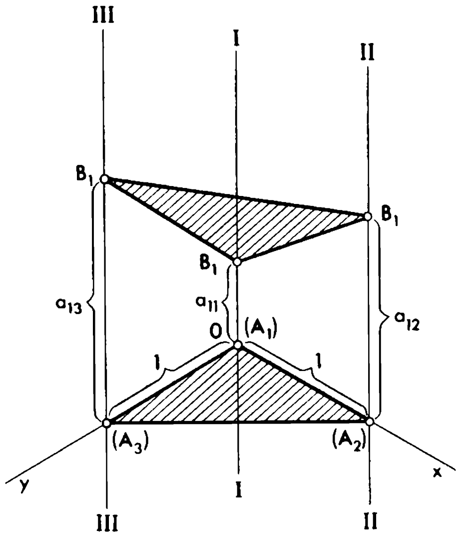

Fig. 5.1

II-II and *III-III* to plot the payoffs for strategies A_2 and A_3. Each of our opponent's strategies, B_j, will be represented by a plane intercepting along the axes *I-I*, *II-II*, and *III-III* segments equal to the payoffs for the corresponding strategies A_1, A_2, A_3, and strategy B_j. By constructing in this way all of our opponent's strategies, we obtain a family of planes over the triangle A_1, A_2, A_3 (Fig. 5.2). It is possible to construct the lower bound

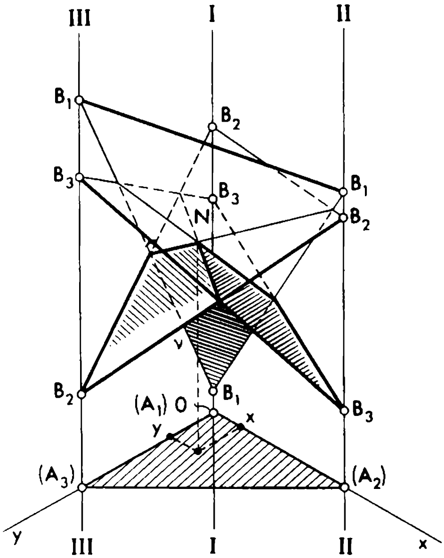

Fig. 5.2

of the payoff for this family, in the same way as we did in the $2 \times n$ case and find the point N on that bound having maximum altitude above the xOy plane. This altitude is the value of the game, ν. The frequencies p_1, p_2, p_3 for the strategies A_1, A_2, A_3 in the optimal strategy S_A^* will be given by the coordinates (x, y) for the point N, namely

$$p_2 = x \quad p_3 = y \quad p_1 = 1 - p_2 - p_3$$

However, such a geometrical representation is not easy to visualize even for the $3 \times n$ case and requires a lot of time and imaginative effort. In the general case the representation is in m-dimensional space and our geometrical intuition fails, although in

some cases it may be found helpful to employ geometrical terminology. When solving $m \times n$ games in practice, it is more convenient to use computational analytic methods rather than geometrical analogies, especially as these methods are the only methods suitable for solving problems on computers.

All these methods essentially reduce to solving problems by successive sampling, but the ordering of a sequence of samples makes it possible to construct an algorithm leading to a solution in the most economical way.

Here we shall dwell briefly on a calculation procedure for solving $m \times n$ games – the so-called "linear programming" method.

To this end we shall first give a general statement of the problem of finding the solution of an $m \times n$ game. Given an $m \times n$ game with m strategies, $A_1, A_2, \ldots, A_m$ for player A and n strategies, $B_1, B_2, \ldots, B_n$, for player B, and payoff matrix $\|a_{ij}\|$, it is required to find a solution of the game, i. e. two optimum mixed strategies for players A and B

$$S_A^* = \begin{pmatrix} A_1 & A_2 & & A_m \\ p_1 & p_2 & \ldots & p_m \end{pmatrix} \quad S_B^* = \begin{pmatrix} B_1 & B_2 & \ldots & B_n \\ q_1 & q_2 & \ldots & q_n \end{pmatrix}$$

where $p_1 + \ldots + p_m = 1$; $q_1 + \ldots + q_n = 1$ (some of the numbers p_i and q_j may be equal to zero).

Our optimal strategy S_A^* must assure us a payoff not less than v whatever our opponent's behaviour, and a payoff equal to v if our opponent's behaviour is optimal (strategy S_B^*). Similarly strategy S_B^* must assure our opponent a loss not greater than v whatever our behaviour, and equal to v if our behaviour is optimal (strategy S_A^*).

In this case we do not know the value of the game, v; we shall assume that it is equal to some positive number. This does not violate the generality of the reasoning; to have $v > 0$, it is evidently sufficient for all elements of matrix $\|a_{ij}\|$ to be nonnegative. This can always be attained by adding to elements $\|a_{ij}\|$ a sufficiently large positive value L; then the value of the game will increase by L while the solution will remain unchanged.

Suppose we have chosen our optimal strategy S_A^*. Then our average payoff, when our opponent plays his strategy B_j, will be

$$a_j = p_1 a_{1j} + p_2 a_{2j} + \ldots + p_m a_{mj}$$

Our optimal strategy S_A^* has the property such that, whatever our opponent's behaviour, his strategy assures us a payoff not less than v; consequently, it is impossible for any of the numbers a_j to be less

than v. A number of conditions result:

$$\left.\begin{array}{c} p_1a_{11} + p_2a_{21} + \dots + p_ma_{m1} \geqslant v \\ p_1a_{12} + p_2a_{22} + \dots + p_ma_{m2} \geqslant v \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ p_1a_{1n} + p_2a_{2n} + \dots + p_ma_{mn} \geqslant v \end{array}\right\} \tag{5.1}$$

We divide inequalities (5.1) by a positive value v and denote

$$p_1/v = \xi_1 \quad p_2/v = \xi_2 \quad p_m/v = \xi_m$$

Then conditions (5.1) can be written down as

$$\left.\begin{array}{c} a_{11}\xi_1 + a_{21}\xi_2 + \dots + a_{m1}\xi_m \geqslant 1 \\ a_{12}\xi_1 + a_{22}\xi_2 + \dots + a_{m2}\xi_m \geqslant 1 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{1n}\xi_1 + a_{2n}\xi_2 + \dots + a_{mn}\xi_m \geqslant 1 \end{array}\right\} \tag{5.2}$$

where $\xi_1, \xi_2, \dots, \xi_n$ are nonnegative numbers. Since $p_1 + p_2 + \dots + p_m = 1$, the variables $\xi_1, \xi_2, \dots, \xi_m$ satisfy the condition

$$\xi_1 + \xi_2 + .. + \xi_m = 1/v \tag{5.3}$$

We want to make our safe payoff as large as possible. The right-hand side of equation (5.3) will then assume a minimal value.

Thus the problem of finding the solution of the game is reduced to the following mathematical problem: *determine the nonnegative variables* $\xi_1, \xi_2, \dots, \xi_m$ *satisfying conditions* (5.2) *so that their sum*

$$\Phi = \xi_1 + \xi_2 + \dots + \xi_m$$

should be minimal.

It is usual in solving extremal (maximum and minimum) problems to differentiate the function and set the derivatives to zero. But such a method is useless in this case, since the function Φ to be minimized is linear and its derivatives with respect to all the arguments are equal to unity, i. e. vanish nowhere. Consequently, the maximum of the function is attained somewhere on the boundary of the argument range, which is determined by the requirement for the nonnegativity of the arguments and by conditions (5.2). The method of finding extreme values by differentiation is also useless in cases where, in order to solve the game, the maximum of the lower (or the minimum of the upper) bound of the payoff is determined, as when solving $2 \times n$ games for instance. Indeed, the lower bound is composed of line segments and the maximum is

attained on the boundary of the interval or at the point of intersection of line segments rather than at the point where the derivative is equal to zero (there is no such point at all).

To solve such problems, which occur rather often in practice, special *linear programming techniques* have been developed by mathematicians.

A linear programming problem is set as follows.

Given a system of linear equations

$$\left.\begin{aligned} a_{11}\xi_1 + a_{21}\xi_2 + \ldots + a_{m1}\xi_m &= b_1 \\ a_{12}\xi_1 + a_{22}\xi_2 + \ldots + a_{m2}\xi_m &= b_2 \\ \cdot \quad \cdot \quad \cdot \qquad\qquad \cdot \quad \cdot \quad & \cdot \\ a_{1n}\xi_1 + a_{2n}\xi_2 + \ldots + a_{mn}\xi_m &= b_n \end{aligned}\right\} \tag{5.4}$$

Find nonnegative values for the variables $\xi_1, \xi_2, \ldots \xi_m$ which satisfy conditions (5.4) and at the same time minimize the given homogeneous linear function of the variables $\xi_1, \xi_2, \ldots, \xi_m$ (the linear form):

$$\Phi = c_1\xi_1 + c_2\xi_2 + \ldots + c_m\xi_m$$

It can be easily seen that the game theory problem given previously is a particular case of the linear programming problem with $c_1 = c_2 = \ldots = c_m = 1$.

It may at first sight seem that conditions (5.2) are not equivalent to conditions (5.4), since they contain inequality signs instead of equality signs. It is easy, however, to get rid of the inequality signs by introducing new dummy nonnegative variables $z_1, z_2, \ldots, z_n$ and writing conditions (5.2) as

$$\left.\begin{aligned} a_{11}\xi_1 + a_{21}\xi_2 + \ldots a_{m1}\xi_m - z_1 &= 1 \\ a_{12}\xi_1 + a_{22}\xi_2 + \ldots + a_{m2}\xi_m - z_2 &= 1 \\ \cdot \quad \cdot \quad \cdot \qquad\qquad \cdot \quad & \cdot \\ a_{1n}\xi_1 + a_{2n}\xi_2 + \ldots + a_{mn}\xi_m - z_n &= 1 \end{aligned}\right\} \tag{5.5}$$

The linear form Φ which must be minimized is equal to

$$\Phi = \xi_1 + \xi_2 + \ldots + \xi_m$$

Linear programming methods make it possible to select, by a comparatively small number of successive trials, variables $\xi_1, \xi_2, \ldots, \xi_m$ which satisfy the set requirements. For greater clarity we shall demonstrate the application of these methods by solving some particular games.

EXAMPLE 1. Find the solution of the 3×3 game given in Example 2 of Section 1, with the following matrix:

$A \backslash B$	B_1	B_2	B_3
A_1	2	-3	4
A_2	-3	4	-5
A_3	4	-5	6

To make all the a_{ij} nonnegative, add $L = 5$ to each element of the matrix. We get the matrix

$A \backslash B$	B_1	B_2	B_3
A_1	7	2	9
A_2	2	9	0
A_3	9	0	11

Here the value of the game increases by 5 while the solution remains unchanged.

We determine the optimal strategy S_A^*. Conditions (5.2) are of the form

$$\left.\begin{array}{l} 7\xi_1 + 2\xi_2 + 9\xi_3 \geqslant 1 \\ 2\xi_1 + 9\xi_2 \qquad\quad\;\; \geqslant 1 \\ 9\xi_1 + \qquad\; 11\xi_3 \geqslant 1 \end{array}\right\} \tag{5.6}$$

where $\xi_1 = p_1/v$; $\xi_2 = p_2/v$; $\xi_3 = p_3/v$.

To get rid of the inequality signs, we introduce the dummy variables z_1, z_2, z_3; conditions (5.6) will be written as

$$\left.\begin{array}{l} 7\xi_1 + 2\xi_2 + 9\xi_3 - z_1 = 1 \\ 2\xi_1 + 9\xi_2 \qquad\quad\; - z_2 = 1 \\ 9\xi_1 + \qquad\; 11\xi_3 - z_3 = 1 \end{array}\right\} \tag{5.7}$$

The linear form Φ is of the form

$$\Phi = \xi_1 + \xi_2 + \xi_3$$

and is to be minimized.

If each of B's three strategies is a "utility" one, then each of the three dummy variables, z_1, z_2, z_3, will vanish (i. e. a payoff equal to the value of the game, v, will be attained for each strategy B_j). But for the present we have no grounds for asserting that all the three strategies are "utility" strategies. To check this, we shall try to express the form Φ in terms of the dummy variables z_1, z_2, z_3, and see if, by putting them zero, the minimum of the form can be obtained. To do this we solve equations (5.7) for the dummy variables ξ_1, ξ_2, ξ_3 (i. e. express ξ_1, ξ_2, ξ_3 in terms of the dummy variables z_1, z_2, z_3):

$$\left.\begin{aligned} \xi_1 &= \frac{1}{20} - \frac{99}{80}z_1 + \frac{11}{40}z_2 + \frac{81}{80}z_3 \\ \xi_2 &= \frac{1}{10} + \frac{11}{40}z_1 + \frac{1}{20}z_2 - \frac{9}{40}z_3 \\ \xi_3 &= \frac{1}{20} + \frac{81}{80}z_1 - \frac{9}{40}z_2 - \frac{59}{80}z_3 \end{aligned}\right\} \tag{5.8}$$

Adding ξ_1, ξ_2, ξ_3 we have

$$\Phi = \frac{1}{5} + \frac{1}{20}z_1 + \frac{1}{10}z_2 + \frac{1}{20}z_3 \tag{5.9}$$

In the expression (5.9) the coefficients of all the z's are positive; it means that any increase of z_1, z_2, z_3 above zero will result only in an increase of the value of the form Φ, and we want it to be minimal. Consequently, the values of z_1, z_2, z_3 minimizing the form (5.9) are

$$z_1 = z_2 = z_3 = 0$$

Substituting them into formula (5.9) we find the minimal value of the form Φ:

$$^1/_{\text{v}} = {}^1/_5$$

giving the value of the game

$$\text{v} = 5$$

Substituting the zero values z_1, z_2, z_3 into formulas (5.8), we find

$$\xi_1 = {}^1/_{20} \quad \xi_2 = {}^1/_{10} \quad \xi_3 = {}^1/_{20}$$

or, multiplying them by ν,

$$p_1 = {}^1/_4 \quad p_2 = {}^1/_2 \quad p_3 = {}^1/_4$$

Thus the optimal strategy for A is found:

$$S_A^* = \begin{pmatrix} A_1 & A_2 & A_3 \\ {}^1/_4 & {}^1/_2 & {}^1/_4 \end{pmatrix}$$

i. e. we should write the figure 1 one-fourth of the time, the figure 2 half of the time, and the figure 3 the remaining one-fourth of the time.

Knowing the value of the game $\nu = 5$, it is possible to find our opponent's optimal strategy

$$S_B^* = \begin{pmatrix} B_1 & B_2 & B_3 \\ q_1 & q_2 & q_3 \end{pmatrix}$$

using the familiar methods. To this end we shall employ any two of our "utility" strategies (for example, A_2 and A_3) and write the equations

$$2q_1 + 9q_2 = 5$$

$$9q_1 + 11(1 - q_2 - q_1) = 5$$

thence $q_1 = q_3 = {}^1/_4$; $q_2 = {}^1/_2$. Our opponent's optimal strategy will be the same as ours:

$$S_B^* = \begin{pmatrix} B_1 & B_2 & B_3 \\ {}^1/_4 & {}^1/_2 & {}^1/_4 \end{pmatrix}$$

Now return to the original (untransformed) game. To do this it is only necessary to subtract from the value of the game, $\nu = 5$, the value $L = 5$ which was added to the matrix elements. We obtain the value of the original game $\nu_0 = 0$. Consequently, the optimal strategies for both parties assure an average payoff equal to zero; the game is equally advantageous or disadvantageous to both parties.

EXAMPLE 2. Sports club A has three alternative team lineups, A_1, A_2, and A_3. Club B has also three alternative lineups, B_1, B_2, and B_3. Making an application for a contest, neither of the clubs knows which of the alternatives its opponent will select. The probabilities of club A winning with different alternative lineups of

the clubs' teams, which are approximately known from the experience of past matches, are given by the matrix.

A \ B	B_1	B_2	B_3
A_1	0.8	0.2	0.4
A_2	0.4	0.5	0.6
A_3	0.1	0.7	0.3

Find with what frequency the clubs should use each of the lineups in matches against each other to score the largest average number of victories.

SOLUTION. The lower value of the game is 0.4; the upper value is 0.6; the solution is sought in the range of mixed strategies. In order not to deal with fractions, we multiply all the matrix elements by 10; as a result the value of the game increases by a factor of 10 but the solution remains unchanged. The following matrix is obtained:

A \ B	B_1	B_2	B_3
A_1	8	2	4
A_2	4	5	6
A_3	1	7	3

Conditions (5.5) are of the form

$$\left.\begin{aligned} 8\xi_1 + 4\xi_2 + \xi_3 - z_1 &= 1 \\ 2\xi_1 + 5\xi_2 + 7\xi_3 - z_2 &= 1 \\ 4\xi_1 + 6\xi_2 + 3\xi_3 - z_3 &= 1 \end{aligned}\right\} \tag{5.10}$$

and the minimum condition is

$$\Phi = \xi_1 + \xi_2 + \xi_3 = \min$$

Let us check if all of our opponent's three strategies are "utility" ones. We first suppose that the dummy variables z_1, z_2, z_3 are zero, and, to check this, solve equations (5.10) for ξ_1, ξ_2, ξ_3:

$$\left.\begin{aligned} \xi_1 &= \frac{10}{136} + \frac{27}{136} z_1 + \frac{6}{136} z_2 - \frac{23}{136} z_3 \\ \xi_2 &= \frac{12}{136} - \frac{22}{136} z_1 - \frac{20}{136} z_2 + \frac{54}{136} z_3 \\ \xi_3 &= \frac{8}{136} + \frac{8}{136} z_1 + \frac{32}{136} z_2 - \frac{32}{136} z_3 \end{aligned}\right\} \tag{5.11}$$

giving

$$136\,\Phi = 30 + 13 z_1 + 18 z_2 - 51 z_3 \tag{5.12}$$

Formula (5.12) shows that increasing the variables z_1 and z_2 in comparison with their supposed zero value can only increase Φ, whereas increasing the variable z_3 can decrease Φ. Care must be taken, however, in increasing z_3, lest the variables ξ_1, ξ_2, ξ_3 dependent on z_3 should be made negative in doing so. Therefore, set the variables z_1 and z_2 in the right-hand side of equations (5.11) to be equal to zero, and increase the variable z_3 to the permissible limits (until one of the variables ξ_1, ξ_2, ξ_3 vanishes). It can be seen from the second equality of (5.11) that increasing z_3 is "safe" for the variable ξ_2, for this only makes it increase. As to the variables ξ_1 and ξ_3, they allow z_3 to be increased only to a certain limit. The variable ξ_1 vanishes when $z_3 = {}^{10}/_{23}$; the variable ξ_3 vanishes before this when $z_3 = {}^{1}/_{4}$. Consequently, giving z_3 its maximum permissible value $z_3 = {}^{1}/_{4}$ makes the variable ξ_3 vanish.

To check if the form Φ is minimized when $z_1 = 0$, $z_2 = 0$, $\xi_3 = 0$, express the remaining (nonzero) variables in terms of z_1, z_2, ξ_3 supposed to be zero.

Solving equations (5.10) for ξ_1, ξ_2 and z_3, we get

$$\left.\begin{aligned} \xi_1 &= \frac{1}{32} + \frac{5}{32} z_1 - \frac{4}{32} z_2 + \frac{23}{32} \xi_3 \\ \xi_2 &= \frac{6}{32} - \frac{2}{32} z_1 + \frac{8}{32} z_2 - \frac{54}{32} \xi_3 \\ z_3 &= \frac{8}{32} + \frac{8}{32} z_1 + \phantom{\frac{8}{32}} z_2 - \frac{136}{32} \xi_3 \end{aligned}\right\}$$

giving

$$32\Phi = 7 + 3z_1 + 4z_2 + \xi_3 \tag{5.13}$$

Formula (5.13) shows that any increase of the variables z_1, z_2, ξ_3 above their supposed zero values can only increase the value of the form Φ. Thus the solution of the game is found; it is given by the values

$$z_1 = z_2 = \xi_3 = 0$$

giving

$$\xi_1 = {}^1/_{32} \quad \xi_2 = {}^3/_{16} \quad z_3 = {}^1/_4$$

Substituting into formula (5.13), we find the value of the game, ν:

$$32\Phi = 7 = {}^{32}/_\nu \quad \nu = {}^{32}/_7$$

Our optimal strategy is

$$S_A^* = \begin{pmatrix} A_1 & A_2 \\ {}^1/_7 & {}^6/_7 \end{pmatrix}$$

The "utility" strategies (team lineups A_1 and A_2) should be played with frequencies of ${}^1/_7$ and ${}^6/_7$; lineup A_3 should never be used.

To find our opponent's optimal strategy, we can in the general case proceed as follows. We change the sign of the payoff, add to the matrix elements a constant value L to make them nonnegative, and solve the problem for our opponent in the same way as we did for ourselves. The fact that we already know the value of the game, ν, to some extent simplifies the problem, however. Besides, in this particular case the problem is further simplified by the fact that the solution comprises only two "utility" strategies for our opponent, B_1 and B_2, since the variable z_3 is nonzero and hence when strategy B_3 is played the value of the game is not attained. By choosing any of player A's "utility" strategies, for example A_1, it is possible to find the frequencies q_1 and q_2. To do this we write the equation

$$8q_1 + 2(1 - q_1) = {}^{32}/_7$$

giving

$$q_1 = {}^3/_7 \quad q_2 = {}^4/_7$$

Our opponent's optimal strategy is

$$S_B^* = \begin{pmatrix} B_1 & B_2 \\ {}^3/_7 & {}^4/_7 \end{pmatrix}$$

i. e. our opponent should not use lineup B_3, and lineups B_1 and B_2 should be used with frequencies of $^3/_7$ and $^4/_7$.

Returning to the original matrix, we determine the value of the game

$$v_0 = {}^{32}/_7 : 10 = 0.457$$

This means that with a large number of matches the number of club A's victories will be 0.457 of all matches.

6. Approximation Methods for Solving Games

In practical problems there is often no need to find an exact solution of the game, it is enough to find an approximate solution giving an average payoff close to the value of the game. An approximate estimate of the value of the game v can be obtained from a simple analysis of the matrix and the determination of the lower (α) and upper (β) values of the game. If α and β are close, there is practically no need to search for an exact solution, it being sufficient to select pure minimax strategies. When α and β are not close, it is possible to obtain a practicable solution using numerical methods of solving games, one of which, the *iteration* method, we shall briefly describe here.

The idea of the method is in short as follows. A "mental experiment" is performed in which opponents A and B play their strategies against each other. The experiment consists of a succession of elementary games each having the matrix of a specified game. The experiment begins with our (player A's) choosing arbitrarily one of our strategies, for example, A_i. Our opponent responds to this with his strategy B_j which is the least advantageous to us, i. e. minimizes the payoff obtained when strategy A_i is played. We respond to this move with our strategy A_k which gives the maximum average payoff when our opponent plays strategy B_j. Then it is again our opponent's turn. He responds to our pair of moves, A_i and A_k, with his strategy B_l which gives us the smallest average payoff when those two strategies (A_i and A_k) are played, and so on. At each step of the iterative process each player responds to any move of the other player with a strategy optimal with respect to all his previous moves, which can be regarded as some mixed strategy in which the pure strategies are represented in proportions corresponding to the frequencies with which they are played.

Such a method represents, as it were, a learning-by-experience model, each of the players testing by experience his opponent's mode of behaviour and trying to respond to it in the way most advantageous to himself.

If such a simulation of the learning process is continued long enough, then the average payoff in one pair of moves (elementary game) will tend to the value of the game, and the frequencies $p_1, \ldots, p_m; q_1, \ldots, q_n$ with which the player's strategies occur in this experiment will approach the frequencies determining the optimal strategies.

Calculations show that the convergence of the method is very slow; this is no obstacle, however, for high-speed computers.

We shall illustrate the application of the iteration method by considerating as an example the 3×3 game solved in Example 2 of the previous section.

The game is given by the matrix

$A \backslash B$	B_1	B_2	B_3
A_1	8	2	4
A_2	4	5	6
A_3	1	7	3

Table 6.1 gives the first 18 steps in the iterative process. In the first column are given the serial numbers of elementary games (pairs of moves), n; the second column gives the numbers, i, of the strategies chosen by player A; the next three columns give the "cumulative payoffs" in the first n games when his opponent's strategies B_1, B_2 and B_3 are played. The minimum of these values is underlined. Next come the numbers, j, of the strategies chosen by A's opponent and, correspondingly, the cumulative payoffs in the n games when strategies A_1, A_2, A_3 are played; the maximum of these values has a bar above. The marked values determine either player's choice of a counter strategy. The last three columns give in succession the minimum average payoffs, $\underline{v}$, equal to the minimum cumulative payoffs divided by the number of games, n; the maximum average payoffs, $\bar{v}$, equal to the maximum cumulative payoffs divided by n, and their arithmetical means $v^* = \frac{\underline{v} + \bar{v}}{2}$. As n increases all the values, $\underline{v}$, $\bar{v}$, and v^*, will tend to

Table 6.1

n	i	B_1	B_2	B_3	j	A_1	A_2	A_3	ν	$\bar{\nu}$	ν^*
1	3	$\underline{1}$	7	3	$\overline{1}$	$\overline{8}$	4	1	1	8	4.50
2	1	9	9	$\underline{7}$	3	$\overline{12}$	10	4	3.50	6.00	4.75
3	1	17	$\underline{11}$	11	2	14	$\overline{15}$	11	3.67	5.00	4.33
4	2	21	$\underline{16}$	17	2	16	$\overline{20}$	18	4.00	5.00	4.50
5	2	25	$\underline{21}$	23	2	18	$\overline{25}$	25	4.20	5.00	4.60
6	2	29	$\underline{26}$	29	2	20	30	$\overline{32}$	4.33	5.33	4.82
7	3	$\underline{30}$	33	32	1	28	$\overline{34}$	33	4.29	4.86	4.57
8	2	$\underline{34}$	38	38	1	36	$\overline{38}$	34	4.25	4.75	4.50
9	2	$\underline{38}$	43	44	1	$\overline{44}$	42	35	4.23	4.89	4.56
10	1	46	$\underline{45}$	48	2	46	$\overline{47}$	42	4.50	4.70	4.60
11	2	$\underline{50}$	50	54	1	$\overline{54}$	51	43	4.55	4.91	4.72
12	1	58	$\underline{52}$	58	2	56	$\overline{56}$	50	4.33	4.66	4.49
13	2	62	$\underline{57}$	64	2	58	$\overline{61}$	57	4.38	4.70	4.54
14	2	66	$\underline{62}$	70	2	60	$\overline{66}$	64	4.43	4.71	4.56
15	2	70	$\underline{67}$	76	2	62	71	$\overline{71}$	4.47	4.73	4.60
16	3	$\underline{71}$	74	79	1	70	$\overline{75}$	72	4.44	4.69	4.56
17	2	$\underline{75}$	79	85	1	78	79	73	4.41	4.65	4.53
18	2	$\underline{79}$	84	91	1	$\overline{86}$	83	74	4.39	4.78	4.58
		...	...	...	...	...					

the value of the game, v, but the value v* will naturally tend to it comparatively more rapidly.

As can be seen from the example, the convergence of the iterations is very slow, but all the same even such a small computation enables one to find an approximate value of the game and bring to light the predominance of "utility" strategies. The value of the method considerably increases when computers are used.

The advantage of the iteration method of solving games is that the amount and complexity of calculations increase comparatively weakly as the numbers of strategies, m and n, increase.

7. Methods for Solving Some Infinite Games

An infinite game is a game in which at least one of the parties has an infinite set of strategies. General methods for solving such games are still weakly developed. Some particular cases allowing comparatively simple solutions may be of interest in applications, however.

Consider a game between two opponents, A and B, each having an infinite (uncountable) set of strategies; for player A these strategies correspond to different values of a continuously varying parameter x, and for B, to those of a parameter y. In this case, instead of being given by matrix $\|a_{ij}\|$ the game is defined by some function of two continuously varying arguments, $a(x, y)$, which we shall call a *gain function* (notice that the function $a(x, y)$ itself need not be continuous). The function $a(x, y)$ can be represented geometrically by some surface $a(x, y)$ over the range of the arguments (x, y) (Fig. 7.1).

The analysis of a gain function $a(x, y)$ is done in a way similar to the analysis of a payoff matrix. One first finds the lower value of the game, α; to this end one determines for each x the minimum of the function $a(x, y)$ over all y:

$$\min_{y} a(x, y)$$

then the maximum of these values over all x (maximin) is sought:

$$\alpha = \max_{x} \min_{y} a(x, y)$$

The upper value of the game (the minimax) is determined in a similar way:

$$\beta = \min_{y} \max_{x} a(x, y)$$

Consider the case when $\alpha = \beta$. Since the value of the game, ν, always lies between α and β, it is their common value, that is ν.

The equality $\alpha = \beta$ implies that the surface $a(x, y)$ has a *saddle point*, i. e. a point with coordinates x_0, y_0 at which $a(x, y)$ is simultaneously minimal with respect to y and maximal with respect to x (Fig. 7.2).

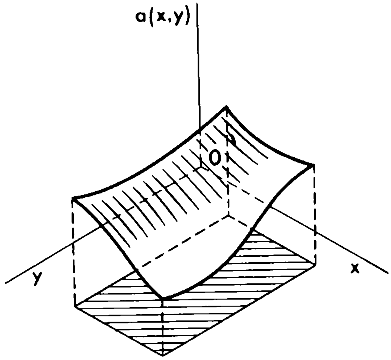

Fig. 7.1

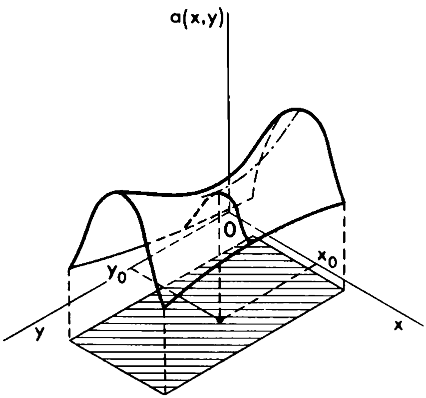

Fig. 7.2

The value of $a(x, y)$ at that point is precisely the value of the game, ν:

$$\nu = a(x_0, y_0)$$

The presence of a saddle point means that the given infinite game has a solution in the range of pure strategies; x_0, y_0 are optimum pure strategies for A and B. In the general case where $\alpha \neq \beta$ the game may have a solution (possibly not a unique one) only in the range of mixed strategies. A mixed strategy for infinite games is some probability distribution for the strategies x and y regarded as random variables. This distribution may be continuous and determined by the densities $f_1(x)$ and $f_2(y)$; or it may be discrete, in which case the optimal strategies consist of a set of separate pure strategies chosen with some nonzero probabilities.

When there is no saddle point in the infinite game, it is possible to give a graphic geometrical interpretation to the lower and the upper value of the game. Consider an infinite game with a gain function $a(x, y)$ and strategies x, y filling continuously the segments of the axes (x_1, x_2) and (y_1, y_2). To determine the lower value of the game, α, it is necessary to "look" at the surface $a(x, y)$ from the y-axis, i. e. project it on the xOa plane (Fig. 7.3). We get some figure bounded by the straight lines

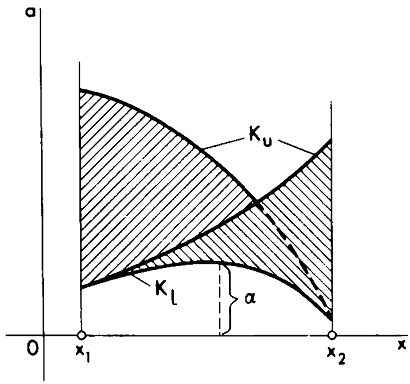

Fig. 7.3

$x = x_1$ and $x = x_2$ at the sides and by the curves K_u and K_l from above and below. The lower value of the game, α, is evidently the maximum ordinate of the curve K_l. Similarly, to find the upper value of the game, β, it is necessary to "look" at the surface $a(x, y)$ from the Ox-axis (to project the surface on the yOa plane) and to find the minimum ordinate of the upper bound K_u of the projection (Fig. 7.4).

Consider two elementary examples of infinite games.

EXAMPLE 1. Players A and B have each an uncountable set of possible strategies x and y, with $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$.

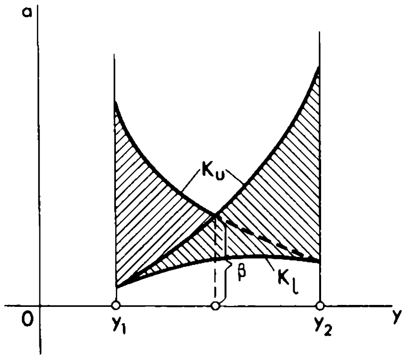

Fig. 7.4

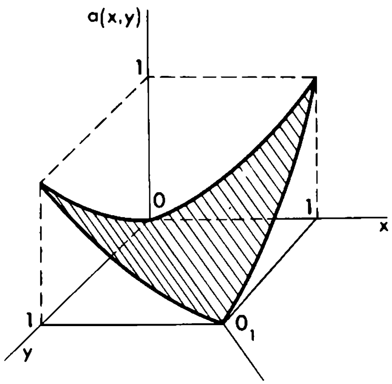

Fig. 7.5

The gain function is given by the expression

$$a(x, y) = (x - y)^2$$

Find the solution of the game.

SOLUTION. The surface $a(x, y)$ is a parabolical cylinder (Fig. 7.5), and there is no saddle point in it. Determine the lower value of the game; evidently, $\min\limits_{y} a(x, y) = 0$ for all x, hence

$$\alpha = \max_{x} \min_{y} a(x, y) = 0$$

Determine the upper value of the game. To this end find, with y fixed,

$$\max_{x} (x - y)^2$$

In this case the maximum is always attained on the boundary of the interval (with $x = 0$ or $x = 1$), i. e. it is equal to the greater of the values y^2, $(1 - y)^2$. Represent graphs of these functions (Fig. 7.6), i. e. the projection of the surface $a(x, y)$ on the yOa

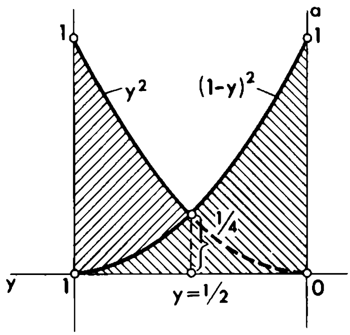

Fig. 7.6

plane. The bold line of Fig. 7.6 shows the function $\max_{x} (x - y)^2$. Its minimal value is evidently attained at $y - 1/2$ and is equal to $1/4$. Consequently, the upper value of the game $\beta = 1/4$.

In this case the upper value of the game coincides with the value of the game, v. Indeed, player A can employ the mixed strategy $S_A = \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix}$ into which the extreme values $x = 0$ and $x = 1$ enter with equal frequencies; then, whatever player B's strategy y, player A's average payoff will be

$$\frac{1}{2} y^2 + \frac{1}{2} (1 - y)^2$$

It can easily be seen that this expression at any values of y

between 0 and 1 has a value not less than $^1/_4$:

$$\frac{1}{2}y^2 + \frac{1}{2}(1-y)^2 \geqslant {}^1/_4$$

Thus, by playing this mixed strategy player A can assure himself a payoff equal to the upper value of the game; since the value of the game cannot be greater than the upper value, this strategy S_A is optimal:

$$S_A = S_A^*$$

It remains to find player B's optimal strategy.

It is obvious that, if the value of the game, ν, is equal to the upper value of the game, β, then player B's optimal strategy will always be his pure minimax strategy assuring him the upper value of the game. Such a strategy in this case is $y_0 = {}^1/_2$. Indeed, when this strategy is played, no matter what player A does, his payoff will not be greater than $^1/_4$. This follows from the obvious inequality

$$(x - {}^1/_2)^2 = x(x-1) + {}^1/_4 \leqslant {}^1/_4$$

EXAMPLE 2. Side A ("we") are shooting at the enemy's aircraft B. To evade the fire the enemy's pilot may manoeuvre with g-load y to which he can arbitrarily assign a value from $y = 0$ (rectilinear motion) to $y = y_{max}$ (flight along a circle of maximum curvature). Assume y_{max} to be a unit of measurement, i. e. set $y_{max} = 1$.

In the fight against the enemy's aircraft we may use some sighting devices based on one or another hypothesis about the motion of the target during the flight of the projectile. The g-load x for this hypothetical manoeuvre can be set equal to any value from 0 to 1.

Our task is to hit the enemy; the enemy's task is to remain unhit. The probability of hitting for the given x and y can be approximated by the formula

$$a(x, y) = pe^{-k(x-y)2}$$

where y is the g-load used by the enemy; x is the g-load allowed for in the sighting device.

Determine the optimal strategies for both sides.

SOLUTION. The solution of the game will evidently remain unchanged, if we put $p = 1$. The gain function $a(x, y)$ is represented by the surface of Fig. 7.7. It is a cylindrical surface whose elements are parallel to the bisector of the coordinate angle

xOy and whose section by a plane perpendicular to an element is a curve of the normal distribution type.

Using the geometrical interpretation proposed above for the lower and the upper value of the game, we find $\beta = 1$ (Fig. 7.8) and $\alpha = e^{-k/4}$ (Fig. 7.9).

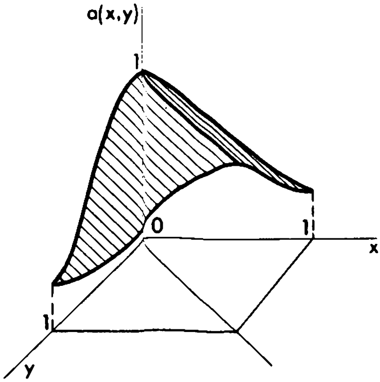

Fig. 7.7

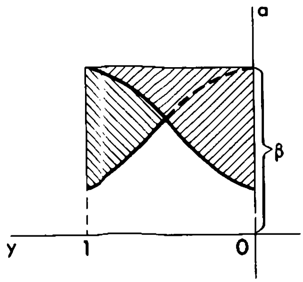

Fig. 7.8

There is no saddle point in the game; the solution must be sought in the range of mixed strategies. The problem is to an extent similar to that of the previous example. Indeed, at low values of k the function $e^{-k(x-y)^2}$ behaves approximately the way the function $-(x-y)^2$ does. and the solution of the game will be obtained if in the solution of the previous example players A and B are made to exchange their roles, i. e. our optimal strategy is the pure

strategy $x = {}^1/_2$, and the enemy's strategy

$$S_B^* = \begin{pmatrix} 0 & 1 \\ {}^1/_2 & {}^1/_2 \end{pmatrix}$$

is to employ the extreme strategies $y = 0$ and $y = 1$ with equal frequencies. This means that we should all the time use the sight

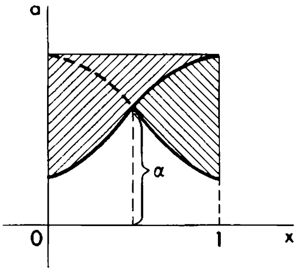

Fig. 7.9

designed for the g-load $x = {}^1/_2$, and the enemy should half of the time not use any manoeuvre at all and half of the time use the maximum permissible manoeuvre.

It can easily be proved that this solution will be valid for the values of $k \leqslant 2$. Indeed, the average payoff when the opponent plays his strategy

$$S_B = \begin{pmatrix} 0 & 1 \\ {}^1/_2 & {}^1/_2 \end{pmatrix}$$

and we play our strategy x is expressed by the function

$$a(x) = \frac{1}{2}(e^{-kx^2} + e^{-k(1-x)^2})$$

which for the values of $k \leqslant 2$ has one maximum equal to the lower value of the game, α, when $x = {}^1/_2$. Consequently, playing strategy S_B assures the enemy a loss not greater than α, from which it is clear that it is α, the lower value of the game, that is the value of the game, ν.

When $k > 2$ the function $a(x)$ has two maxima (Fig. 7.10) situated symmetrically with respect to $x = {}^1/_2$ at the points x_0 and $1 - x_0$, the value of x_0 depending on k.

It is obvious that when $k = 2$ $x_0 = 1 - x_0 = {}^1/_2$; as k is increased, the points x_0 and $1 - x_0$ move apart tending towards the extreme

points (0 and 1). Consequently, the solution of the game will depend on k. Prescribe a specific value to k, for instance $k = 3$, and find the solution of the game. To this end, determine the abscissa x_0 of the maximum of the curve $a(x)$.

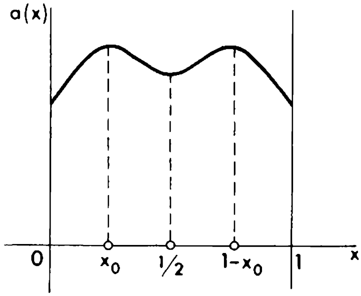

Fig. 7.10

Equating the derivative of the function $a(x)$ to zero, write the following equation to determine x_0:

$$xe^{-3x^2} = (1 - x)e^{-3(1-x)^2}$$

The equation has three roots; $x = {}^1/_2$ (where the minimum is attained) and x_0, $1 - x_0$ where the maxima are attained. Solving the equation numerically, we find approximately

$$x_0 \approx 0.07$$

$$1 - x_0 \approx 0.93$$

We shall prove that in this case the solution of the game is the following pair of strategies:

$$S_A^* = \begin{pmatrix} x_0 & 1 - x_0 \\ {}^1/_2 & {}^1/_2 \end{pmatrix}$$

$$S_B^* = \begin{pmatrix} 0 & 1 \\ {}^1/_2 & {}^1/_2 \end{pmatrix}$$

When we play our strategy S_A^* and the enemy plays his strategy y the average payoff is

$$a_1(y) = \frac{1}{2}\left(e^{-3(0.07-y)^2} + e^{-3(0.93-y)^2}\right)$$

Let us find the minimum of $a_1(y)$ when $0 < y < 1$. The function $a_1(y)$ is symmetrical in relation to $y = {}^1/_2$ and can have only one or two maxima; its minimum is in any case attained either in the middle of the interval (0, 1) or at its ends. Putting $y = 0$ we find

$$a_1(0) = a_1(1) = \frac{1}{2}(e^{-3 \times 0.07^2} + e^{-3 \times 0.93^2}) = 0.530$$

Putting $y = {}^1/_2$ we get

$$a_1({}^1/_2) = e^{-3 \times 0.43^2} = 0.574$$

which is greater than $a_1(0)$; consequently, the value of the game is not less than $a_1(0)$:

$$\nu \geqslant \frac{1}{2}(e^{-3x_0^2} + e^{-3(1 - x_0^2)}) = 0.530$$

Now suppose that the enemy plays strategy S_B^* and we play the strategy x. Then the average payoff will be

$$a_2(x) = \frac{1}{2}(e^{-3x^2} + e^{-3(1 - x)^2}) \tag{7.1}$$

But we have chosen x_0 exactly so that for $x = x_0$ the maximum of the expression (7.1) should be attained; consequently,

$$a_2(x) \leqslant \frac{1}{2}(e^{-3x_0^2} + e^{-3(1 - x_0)^2}) = 0.530$$

i. e. by employing strategy S_B^* the enemy can avoid a loss greater than 0.530; consequently, $\nu = 0.530$ is exactly the value of the game and strategies S_A^* and S_B^* give the solution. This means that we should use the sights with $x = 0.07$ and $x = 0.93$ with the same frequency, and the enemy should with the same frequency not manoeuvre and manoeuvre with the maximum g-load.

Notice that the payoff $\nu = 0.530$ is appreciably greater than the lower value of the game

$$\alpha = e^{-k/4} = e^{-0.75} = 0.472$$

which we could assure ourselves by playing our maximin strategy $x_0 = {}^1/_2$.

One of the practical ways of solving infinite games is through approximately reducing them to finite ones. In doing so a whole range of possible strategies for each player is arbitrarily combined into one strategy. Of course, one can obtain only an approximate solution for a game in this way, but in most cases no exact solution is required.

One should bear in mind, however, that the use of this method may lead to solutions in the range of mixed strategies even in the cases where the solution of the original infinite game is possible in pure strategies, i. e. where the infinite game has a saddle point. If reducing an infinite game to a finite one yields a mixed solution which contains only two neighbouring "utility" strategies, then it is worthwhile attempting the in-between pure strategy of the original infinite game.

Notice in conclusion that unlike finite games, infinite games must not necessarily have a solution. We shall cite an example of an infinite game which has no solution. Two players name each an integral number. The player who names a larger number gets a dollar from the other. If both name the same number, the game ends in a draw. The game evidently cannot have a solution. There are classes of infinite games, however, for which there is clearly a solution. It is possible to prove, in particular, that if in an infinite game possible strategies x, y for players A and B fill continuously some intervals and the gain function $a(x, y)$ is continuous, then there is always a solution to the game (in pure or mixed strategies).

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A. SMOGORZHEVSKII, D. Sc.

The booklet deals with a fundamental method of analytical geometry and its ability to describe geometrical figures through equations. This method permits geometrical study and solution of geometrical problems by algebraic means, thus making a visual representation of complicated spatial configurations superfluous. Considerable attention has been devoted to the question of representing geometrical figures through equations, which is often difficult for those who begin to study the method of coordinates. Detailed examples havê been given to illustrate the application of the method.

The booklet is intended for senior schoolboys and all those interested in mathematics.

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A. MARKUSHEVICH, Mem. USSR Acad. Sc.

This small booklet has gained considerable popularity among mathematics lovers. It is based on a lecture delivered by the author to Moscow schoolboys of 7th and 8th forms, and contains a description of a circle, ellipse, hyperbola, parabola, Archimedean spiral, and other curves. The book has been revised and enlarged several times in keeping with the demands from readers, but the characteristic style of the book, which is demonstrative and lucid rather than deductive and dry, has been retained.

The booklet is intended for those who are interested in mathematics and possess a middle standard background.

This booktreats in a popular manner the elements of game theory and some methods for solving matrix games. It contains almost no proofs and illustrates the basic principles with examples. To be able to read the book, an acquaintance with the elements of probability theiry and calculus is enough. The book is intended to disseminate the ideas of game theory which have practical economic or military applications.

Mir Publishers
Moscow